Four Easy Pieces – Explicit R Matrices from the $(\dot{0}_m|\alpha)$ Highest Weight Representations of $U_q[gl(m|1)]$

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Abstract

We provide explicit presentations of members of a suite of R matrices arising from the $(\dot{0}_m|\alpha)$ representations of the quantum superalgebras $U_q[gl(m|1)]$. Our algorithm constructs both trigonometric and quantum R matrices; all of which are *graded*, in that they solve a graded Yang–Baxter equation. This grading is easily removed, yielding R matrices that solve the usual Yang–Baxter equation. For $m > 2$, the computations are impracticable for a human to perform, so we have implemented the entire process in MATHEMATICA, and then performed the computations for $m = 1, 2, 3$ and 4.

1 Overview

This paper describes the results of the automation of an algorithm to explicitly generate several R matrices. Specifically, we construct trigonometric R matrices $\check{R}^m(u)$ corresponding to the α-parametric highest weight minimal representations labeled $(\dot{0}_m|\alpha)$, of the quantum superalgebras $U_q[gl(m|1)]$. These representations are 2^m dimensional, irreducible, and contain free complex parameters q and α; the real variable u is a spectral parameter. Quantum R matrices $\check{R}^m$ are immediately obtainable as the spectral limits $u \to \infty$ of the $\check{R}^m(u)$.

Our R matrices are in fact *graded*, as they are based on graded vector spaces, hence they actually satisfy graded Yang–Baxter equations. However, it is a simple matter to remove this grading and transform them into objects that satisfy the usual Yang–Baxter equations.

These R matrices are of physical interest in that they are applicable to the construction of exactly solvable models of interacting fermions. Corresponding to $\check{R}^m(u)$, we may construct an integrable 2^m state fermionic model on a lattice. Models associated with $m = 2$ and $m = 3$ have been discussed in [9] and [8], respectively. The $m = 4$ case has an elegant interpretation in terms of a 2-leg ladder model for interacting electrons: a discussion of this is provided in §5.

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Furthermore, from each of our $\check{R}^m$, we may obtain a two variable polynomial 'Links–Gould' link invariant LG^m [12]. LG^1 degenerates to being the Alexander–Conway polynomial in the single variable $q^{2\alpha}$ (c.f. [1]). LG^2 is in fact more powerful than the well known two variable HOMFLY and Kauffman invariants, although it cannot distinguish mutants or inversion [4, 6]. LG^m for $m > 2$ have similar gross properties to LG^2, although they should be able to distinguish more links [5].

The process has been implemented in Mathematica, and R matrices computed for $m = 1, 2, 3$ and 4. The description of the computational details of the algorithms used to construct the R matrices is rather long, and will be provided elsewhere [3], as will observations relating to the construction of the new invariants [5].

2 Algebraic Details

Fixing m, we are initially interested in a 2^m dimensional vector space V that is a module for the $U_q[gl(m|1)]$ representation $\Lambda = (\dot{0}_m|\alpha)$. The algebra contains a free complex variable q, whilst the representation π_Λ acting on V contains a free complex variable α. Our V is actually ($\mathbb{Z}_2$) *graded*; this ensures compatibility with the ($\mathbb{Z}_2$) grading of $U_q[gl(m|1)]$. A full description of $U_q[gl(m|n)]$ in terms of generators and relations is contained in [13, pp1237-1238]; for our purposes a set of *simple* generators for $U_q[gl(m|1)]$ is:

$$\left\{ \begin{array}{lll} K_a, & a = 1, \ldots, m+1 & \text{(Cartan)}, \\ E^a{}_{a+1}, & a = 1, \ldots, m & \text{(raising)}, \\ E^{a+1}{}_a, & a = 1, \ldots, m & \text{(lowering)} \end{array} \right\}.$$

We apply the Kac induced module construction (KIMC) [11] to establish a *weight* basis $\{v_i\}_{i=1}^{2^m}$ for V. This involves postulating v_1 as a highest weight vector, and recursively acting on v_1 with all possible distinct products of simple lowering generators $E^{a+1}{}_a$ to define the other basis vectors, normalising as we go. This construction requires a PBW basis for $U_q[gl(m|1)]$, which enables us to transform any element of the algebra into a normal form ([13], see also [3]).

The tensor product module $V \otimes V$ has a natural (weight) basis $\{v_i \otimes v_j\}_{i,j=1}^{2^m}$, which inherits the grading of V. To build R matrices acting on $V \otimes V$, we require an alternative, orthonormal weight basis B for $V \otimes V$ corresponding to its decomposition into irreducible $U_q[gl(m|1)]$ submodules. Again using the KIMC, the basis vectors of B are derived as linear combinations of the form $\gamma_{ij}(v_i \otimes v_j)$, where the coefficients γ_{ij} are algebraic expressions in q and α. (This process initially yields a basis for each submodule that is not necessarily orthonormal, so we also apply a Gram–Schmidt process.)

For our particular representation, the orthogonal decomposition of $V \otimes V$ contains no multiplicities [7, (34)]:

$$V \otimes V \quad \triangleq \quad \bigoplus_{k=1}^{m+1} V_k,$$

where V_k has highest weight $\lambda_k = (\dot{0}_{m+1-k}, \dot{-1}_{k-1} \,|\, 2\alpha + k - 1)$.

The R matrices are then formed as weighted sums of projectors onto these submodules V_k. Explicitly, where $\check{P}_k$ is the projector onto submodule V_k, we have:

$$\check{R}^m(u) = \sum_{k=1}^{m+1} \Xi_k \check{P}_k, \qquad\qquad \check{R}^m = \sum_{k=1}^{m+1} \xi_k \check{P}_k,$$

where Ξ_k and ξ_k are the following eigenvalues of the R matrices on the submodules V_k ([10], and c.f. [8]):

$$\begin{aligned} \Xi_k &= \prod_{j=0}^{k-2} \frac{[\alpha + j + u]_q}{[\alpha + j - u]_q}, \\ \xi_k &= \lim_{u \to \infty} \Xi_k = (-1)^{k-1} q^{(k-1)(2\alpha+k-2)}, \end{aligned}$$

where we intend $\Xi_1 = \xi_1 = 1$, and we have used the q bracket:

$$[X]_q \triangleq \frac{q^{+X} - q^{-X}}{q^{+1} - q^{-1}}; \qquad \text{observe that} \qquad \lim_{q \to 1} [X]_q = X.$$

Thus we have the intended spectral limit $\check{R}^m = \lim_{u \to \infty} \check{R}^m(u)$. The resulting R matrices are normalised such that the coefficients of the 'first' components (viz e_{11}^{11}) are unity. For applications, other choices of normalisation may be applicable [5].

To be certain, $\check{R}^m(u)$ satisfies the following graded version of the trigonometric Yang–Baxter equation:

$$\begin{aligned} &(-1)^{[b'][c']+[a'][c]+[a][b]+[b'][b'']} \check{R}(u)_{b'c'}^{c''b''} \, \check{R}(u+v)_{a'c}^{c'a''} \, \check{R}(v)_{ab}^{b'a'} \\ &\quad = (-1)^{[a'][b']+[a][c']+[b][c]+[b][b']} \check{R}(v)_{a'b'}^{b''a''} \, \check{R}(u+v)_{ac'}^{c''a'} \, \check{R}(u)_{bc}^{c'b'}, \end{aligned} \tag{1}$$

where $[a]$ is the grading of the vector v_a. The parity factors in (1) may be removed by the following transformation (e.g. see [2]):

$$\check{R}_{ab}^{a'b'}(u) \mapsto (-1)^{[a]([b]+[b'])} \check{R}_{ab}^{a'b'}(u),$$

after which $\check{R}(u)$ which satisfied (1) now satisfies the usual ungraded TYBE:

$$\check{R}(u)_{b'c'}^{c''b''} \, \check{R}(u+v)_{a'c}^{c'a''} \, \check{R}(v)_{ab}^{b'a'} = \check{R}(v)_{a'b'}^{b''a''} \, \check{R}(u+v)_{ac'}^{c''a'} \, \check{R}(u)_{bc}^{c'b'}, \tag{2}$$

written in noncomponent form as:

$$\check{R}_{12}(u) \check{R}_{23}(u+v) \check{R}_{12}(v) = \check{R}_{23}(v) \check{R}_{12}(u+v) \check{R}_{23}(u). \tag{3}$$

In the spectral limit $\check{R} = \lim_{u\to\infty} \check{R}(u)$, this of course becomes a QYBE:

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \tag{4}$$

viz $(\check{R}\otimes I)(I\otimes\check{R})(\check{R}\otimes I) = (I\otimes\check{R})(\check{R}\otimes I)(I\otimes\check{R})$, familiar as the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

Defining $R(u) \triangleq P\check{R}(u)$, where P is a permutation operator, yields a trigonometric R matrix $R(u)$ satisfying the following version of (3):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{5}$$

This transformation amounts to the mapping: $R(u)_{ab}^{a'b'} = \check{R}(u)_{ab}^{b'a'}$. In component form, (5) is more symmetric than (2):

$$R(u)_{b'c'}^{b''c''}R(u+v)_{a'c}^{a''c'}R(v)_{ab}^{a'b'} = R(v)_{a'b'}^{a''b''}R(u+v)_{ac'}^{a'c''}R(u)_{bc}^{b'c'}.$$

3 Implementation

The entire process has been implemented as a suite of functions in the interpreted environment of MATHEMATICA. Whilst there is no theoretical limit to m, storage and patience mean that a current reasonable practical limit for m is 4. The computations are computationally inefficient! Translation of the several thousand lines of MATHEMATICA code into a compiled language would increase the speed of the algorithm enormously, but storage requirements would still limit m.

4 Results

Both $\check{R}^m(u)$ and $\check{R}^m$ have been obtained for $m = 1,2,3,4$. Of these, the $m = 1$ case (c.f. [1]), can be done by hand in a couple of hours; the complete $m = 2$ case appears in my PhD thesis [2], and took several weeks to do by hand; partial details of the $m = 3$ case appear in [8]; whilst the $m = 4$ case is new. By direct substitution, we have been able to verify that each $\check{R}^m(u)$ satisifes (3)[1] and that each $\check{R}^m$ satisfies (4).

Fixing m, each R matrix contains 2^{4m} (albeit mostly zero) components. Let N'_m and N_m be the number of nonzero components of $\check{R}^m(u)$ and $\check{R}^m$ respectively. As $\check{R}^m$ is the spectral limit of $\check{R}^m(u)$, we expect $N_m \leqslant N'_m$. The numbers of nonzero components of each of the $m+1$ projectors are similar to N_m and N'_m; and $N'_m = 6^m$ (why?). Let $s_m \triangleq N_m/2^{4m}$ be the sparsity of $\check{R}^m$. Table 1 summarises our results.

Listings of the nonzero components of our R matrices are supplied in the Appendix.

[1]We didn't check $\check{R}^4(u)$, as the computations would have been excessively expensive.

m	2^{4m}	projector sizes	N'_m	N_m	$s_m(\%)$
1	16	$5, 5$	6	5	31.3
2	256	$25, 34, 25$	36	26	10.2
3	4096	$125, 199, 199, 125$	216	139	3.4
4	65536	$625, 1124, 1254, 1124, 625$	1296	758	1.1

Table 1: Numbers of nonzero components of our R matrices.

5 An Application

Of particular new interest is the interpretation of our $U_q[gl(4|1)]$ trigonometric R matrix $\check{R}^4(u)$ in the construction of an exactly solvable 2-leg ladder model of interacting electrons. To this end, consider a 2-leg ladder, with electron occupation sites at the end of each rung. Each site may contain a maximum of 2 electrons, each in state spin up $\uparrow$ or down $\downarrow$. Thus, at each site we have 4 possible states: unoccupied $|0\rangle$, both up $|\uparrow\uparrow\rangle$, both down $|\downarrow\downarrow\rangle$, or mixed $|\uparrow\downarrow\rangle$. Taken together, each rung space, corresponding to our V, is 16 dimensional. These 16 dimensions correspond to 1 electron-free state, 4 single-electron states, 6 two-electron states, 4 three-electron states and 1 four-electron state.

A Hamiltonian determined by $\check{R}^4(u)$ describes the interactions between rungs. Discernment of the details of the terms contained within this Hamiltonian are left as an exercise for the reader with some idle time; the procedure essentially follows [1, 8, 9].

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Appendix

Below, we list the nonzero components of our graded R matrices. The data are presented in terms of elementary rank 4 tensors e^{ik}_{jl}, obtained by inserting a copy of e^k_l at each location of e^i_j; where the e^i_j are elementary $2^m \times 2^m$ matrices. We also use the following notation:

- To increase literacy, we replace $[X]_q$ with $[X]$, we often substitute $\overline{q}$ for q^{-1}, and we set $\Delta \triangleq q - \overline{q}$.
- To convert these *graded* R matrices to the equivalent ungraded objects, simply multiply all terms in **boldface** by -1.
- The following notation is a convenient shorthand for the frequently appearing *q graded symmetric combination* of rank 4 tensors:

  $$q^x_\pm e^{ik}_{jl} \quad \triangleq \quad q^x e^{ik}_{jl} \pm \overline{q}^x e^{ki}_{lj}.$$

  Not using it allows us to present both graded and grading-stripped R matrices in one unit.

R matrices for m = 1

Here, $[1] = 0$ and $[2] = 1$, and $\check{R}^1(u)$ has 6 nonzero components:

$$1\left\{e^{11}_{11}\right\}, \qquad \frac{[\alpha+u]}{[\alpha-u]}\left\{e^{22}_{22}\right\}, \qquad \frac{[\alpha]}{[\alpha-u]}\begin{Bmatrix} \overline{q}^u \left\{e^{12}_{12}\right\} \\ q^u \left\{e^{21}_{21}\right\} \end{Bmatrix}, \qquad \frac{[u]}{[\alpha-u]}\begin{Bmatrix} +\mathbf{1}\left\{e^{12}_{21}\right\} \\ -1\left\{e^{21}_{12}\right\} \end{Bmatrix}.$$

$\check{R}^1$ has 5 nonzero components:

$$1\left\{e^{11}_{11}\right\}, \qquad -q^{2\alpha}\left\{e^{22}_{22}\right\}, \qquad -\Delta q^\alpha[\alpha]\left\{e^{21}_{21}\right\}, \qquad q^\alpha \begin{Bmatrix} -\mathbf{1}\left\{e^{12}_{21}\right\} \\ +1\left\{e^{21}_{12}\right\} \end{Bmatrix}.$$

R matrices for $\mathbf{m=2}$

Here, $[1]=[4]=0$ and $[2]=[3]=1$, and $\check{R}^2(u)$ has 36 nonzero components:

$$1\left\{e^{11}_{11}\right\}, \qquad \frac{[\alpha+u]}{[\alpha-u]}\left\{e^{22}_{22},e^{33}_{33}\right\}, \qquad \frac{[\alpha+u][\alpha+1+u]}{[\alpha-u][\alpha+1-u]}\left\{e^{44}_{44}\right\},$$

$$\frac{[\alpha]}{[\alpha-u]}\begin{Bmatrix} \overline{q}^{u}\left\{e^{12}_{12},e^{13}_{13}\right\} \\ q^{u}\left\{e^{21}_{21},e^{31}_{31}\right\}\end{Bmatrix}, \qquad \frac{[\alpha+1][\alpha+u]}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} q^{u}\left\{e^{43}_{43},e^{42}_{42}\right\} \\ \overline{q}^{u}\left\{e^{34}_{34},e^{24}_{24}\right\}\end{Bmatrix},$$

$$\frac{[\alpha][\alpha+1]}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} \overline{q}^{2u}\left\{e^{14}_{14},\right\} \\ q^{2u}\left\{e^{41}_{41}\right\}\end{Bmatrix}, \quad \frac{1}{\Delta^2[\alpha-u][1+\alpha-u]}\begin{Bmatrix} f(\overline{q})\left\{e^{23}_{23}\right\} \\ f(q)\left\{e^{32}_{32}\right\}\end{Bmatrix},$$

$$\frac{[u]}{[\alpha-u]}\begin{Bmatrix} +\mathbf{1}\left\{e^{12}_{21},e^{13}_{31}\right\} \\ -1\left\{e^{21}_{12},e^{31}_{13}\right\}\end{Bmatrix}, \qquad \frac{[u][\alpha+u]}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} -\mathbf{1}\left\{e^{43}_{34},e^{42}_{24}\right\} \\ +1\left\{e^{34}_{43},e^{24}_{42}\right\}\end{Bmatrix},$$

$$\frac{[u-1][u]}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} e^{14}_{41} \\ e^{41}_{14}\end{Bmatrix}, \qquad -\frac{[u]^2}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} e^{23}_{32} \\ e^{32}_{23}\end{Bmatrix},$$

$$\frac{[\alpha]^{\frac{1}{2}}[\alpha+1]^{\frac{1}{2}}[u]}{[\alpha-u][\alpha+1-u]}\begin{Bmatrix} q^{u}\left\{q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{e}^{\mathbf{41}}_{\mathbf{32}} \\ +e^{32}_{41}\end{Bmatrix}, \overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{e}^{\mathbf{41}}_{\mathbf{23}} \\ -e^{23}_{41}\end{Bmatrix}\right\} \\ \overline{q}^{u}\left\{\overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{e}^{\mathbf{14}}_{\mathbf{23}} \\ -e^{23}_{14}\end{Bmatrix}, q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{e}^{\mathbf{14}}_{\mathbf{32}} \\ +e^{32}_{14}\end{Bmatrix}\right\}\end{Bmatrix},$$

where $f(q) = -2q + q^{2u}(q-\overline{q}) + q^{2\alpha}(q+\overline{q})$.

$\check{R}^2$ has 26 nonzero components:

$$1\left\{e^{11}_{11}\right\}, \qquad -q^{2\alpha}\left\{e^{22}_{22},e^{33}_{33}\right\}, \qquad q^{4\alpha+2}\left\{e^{44}_{44}\right\},$$

$$-\Delta q^{\alpha}[\alpha]\left\{e^{21}_{21},e^{31}_{31}\right\}, \quad \Delta q^{3\alpha+1}[\alpha+1]\left\{e^{43}_{43},e^{42}_{42}\right\}, \quad \Delta^2 q^{2\alpha+1}[\alpha][\alpha+1]\left\{e^{41}_{41}\right\}, \quad \Delta q^{2\alpha+1}\left\{e^{32}_{32}\right\},$$

$$q^{\alpha}\begin{Bmatrix} -\mathbf{1}\left\{e^{12}_{21},e^{13}_{31}\right\} \\ +1\left\{e^{21}_{12},e^{31}_{13}\right\}\end{Bmatrix}, \qquad q^{3\alpha+1}\begin{Bmatrix} -\mathbf{1}\left\{e^{43}_{34},e^{42}_{24}\right\} \\ +1\left\{e^{34}_{43},e^{24}_{42}\right\}\end{Bmatrix}, \qquad q^{2\alpha}\begin{Bmatrix} e^{14}_{41} \\ e^{41}_{14}\end{Bmatrix}, \qquad -q^{2\alpha+1}\begin{Bmatrix} e^{23}_{32} \\ e^{32}_{23}\end{Bmatrix},$$

$$\Delta q^{2\alpha+1}[\alpha]^{\frac{1}{2}}[\alpha+1]^{\frac{1}{2}}\left\{q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{e}^{\mathbf{41}}_{\mathbf{32}} \\ +e^{32}_{41}\end{Bmatrix}, \overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{e}^{\mathbf{41}}_{\mathbf{23}} \\ -e^{23}_{41}\end{Bmatrix}\right\}.$$

R matrices for m = 3

Here, $[i]=0$ for $i\in\{1;5,6,7\}$ and $[i]=1$ for $i\in\{2,3,4;8\}$. The reader will have by now appreciated the recurring patterns in the components of our R matrices. To save space, we introduce a little more notation, which eliminates the q brackets altogether:

$$\begin{aligned} S_i^{\pm} &\triangleq [\alpha+i\pm u]_q, \\ A_i^{z} &\triangleq [\alpha+i]_q^{z}, \qquad \text{where } z\in\{\tfrac{1}{2},1\}, \\ U_i^{z} &\triangleq [u-i]_q^{z}, \qquad \text{where } z\in\{1,2\}, \end{aligned}$$

and $i\in\{0,1,\dots,m-1\}$. With this notation, $\check{R}^3(u)$ has 216 nonzero components:

$$1\left\{e_{11}^{11}\right\}, \qquad \frac{S_0^+}{S_0^-}\left\{e_{22}^{22},e_{33}^{33},e_{44}^{44}\right\}, \qquad \frac{S_0^+S_1^+}{S_0^-S_1^-}\left\{e_{55}^{55},e_{66}^{66},e_{77}^{77}\right\}, \qquad \frac{S_0^+S_1^+S_2^+}{S_0^-S_1^-S_2^-}\left\{e_{88}^{88}\right\},$$

$$\frac{A_0}{S_0^-}\left\{\begin{matrix}\overline{q}^u\left\{e_{12}^{12},e_{13}^{13},e_{14}^{14}\right\}\\ q^u\left\{e_{21}^{21},e_{31}^{31},e_{41}^{41}\right\}\end{matrix}\right\}, \qquad \frac{A_2S_0^+S_1^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}q^u\left\{e_{87}^{87},e_{86}^{86},e_{85}^{85}\right\}\\ \overline{q}^u\left\{e_{78}^{78},e_{68}^{68},e_{58}^{58}\right\}\end{matrix}\right\},$$

$$\frac{1}{\Delta^2S_0^-S_1^-}\left\{\begin{matrix}f_1(q)\left\{e_{23}^{23},e_{24}^{24},e_{34}^{34}\right\}\\ f_1(\overline{q})\left\{e_{32}^{32},e_{42}^{42},e_{43}^{43}\right\}\end{matrix}\right\}, \qquad \frac{1}{\Delta^2S_0^-S_1^-S_2^-}\left\{\begin{matrix}f_2(q)\left\{e_{76}^{76},e_{75}^{75},e_{65}^{65}\right\}\\ f_2(\overline{q})\left\{e_{67}^{67},e_{57}^{57},e_{56}^{56}\right\}\end{matrix}\right\},$$

$$\frac{A_0A_1}{S_0^-S_1^-}\left\{\begin{matrix}q^{2u}\left\{e_{51}^{51},e_{61}^{61},e_{71}^{71}\right\}\\ \overline{q}^{2u}\left\{e_{15}^{15},e_{16}^{16},e_{17}^{17}\right\}\end{matrix}\right\}, \qquad \frac{A_1A_2S_0^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}q^{2u}\left\{e_{84}^{84},e_{83}^{83},e_{82}^{82}\right\}\\ \overline{q}^{2u}\left\{e_{48}^{48},e_{38}^{38},e_{28}^{28}\right\}\end{matrix}\right\},$$

$$\frac{A_1}{\Delta^2S_0^-S_1^-S_2^-}\left\{\begin{matrix}f_3(q)\left\{e_{27}^{27}\right\},f_4(q)\left\{e_{36}^{36}\right\},f_5(q)\left\{e_{45}^{45}\right\}\\ f_3(\overline{q})\left\{e_{72}^{72}\right\},f_4(\overline{q})\left\{e_{63}^{63}\right\},f_5(\overline{q})\left\{e_{54}^{54}\right\}\end{matrix}\right\},$$

$$\frac{A_1S_0^+}{S_0^-S_1^-}\left\{\begin{matrix}q^u\left\{e_{52}^{52},e_{53}^{53},e_{62}^{62},e_{64}^{64},e_{73}^{73},e_{74}^{74}\right\}\\ \overline{q}^u\left\{e_{25}^{25},e_{35}^{35},e_{26}^{26},e_{46}^{46},e_{37}^{37},e_{47}^{47}\right\}\end{matrix}\right\}, \qquad \frac{A_0A_1A_2}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}q^{3u}\left\{e_{81}^{81}\right\}\\ \overline{q}^{3u}\left\{e_{18}^{18}\right\}\end{matrix}\right\},$$

$$\frac{U_0}{S_0^-}\left\{\begin{matrix}+\mathbf{1}\left\{e_{21}^{12},e_{31}^{13},e_{41}^{14}\right\}\\ -1\left\{e_{12}^{21},e_{13}^{31},e_{14}^{41}\right\}\end{matrix}\right\}, \qquad \frac{U_0S_0^+S_1^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}+\mathbf{1}\left\{e_{87}^{78},e_{86}^{68},e_{85}^{58}\right\}\\ -1\left\{e_{78}^{87},e_{68}^{86},e_{58}^{85}\right\}\end{matrix}\right\}, \qquad \frac{U_0U_1U_2}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}+\mathbf{1}\left\{e_{81}^{18}\right\}\\ -1\left\{e_{18}^{81}\right\}\end{matrix}\right\},$$

$$\frac{U_0U_1}{S_0^-S_1^-}\left\{\begin{matrix}e_{51}^{15},e_{61}^{16},e_{71}^{17}\\ e_{15}^{51},e_{16}^{61},e_{17}^{71}\end{matrix}\right\}, \qquad -\frac{U_0U_1S_0^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}e_{48}^{84},e_{38}^{83},e_{28}^{82}\\ e_{84}^{48},e_{83}^{38},e_{82}^{28}\end{matrix}\right\},$$

$$-\frac{U_0^2}{S_0^-S_1^-}\left\{\begin{matrix}e_{32}^{23},e_{42}^{24},e_{43}^{34}\\ e_{23}^{32},e_{24}^{42},e_{34}^{43}\end{matrix}\right\}, \qquad -\frac{U_0^2S_0^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}e_{67}^{76},e_{57}^{75},e_{56}^{65}\\ e_{76}^{67},e_{75}^{57},e_{65}^{56}\end{matrix}\right\},$$

$$\frac{U_0^2U_1}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}+\mathbf{1}\left\{e_{45}^{54},e_{36}^{63},e_{27}^{72}\right\}\\ -1\left\{e_{54}^{45},e_{63}^{36},e_{72}^{27}\right\}\end{matrix}\right\}, \qquad \frac{U_0S_0^+}{S_0^-S_1^-}\left\{\begin{matrix}-\mathbf{1}\left\{e_{25}^{52},e_{35}^{53},e_{26}^{62},e_{46}^{64},e_{37}^{73},e_{47}^{74}\right\}\\ +1\left\{e_{52}^{25},e_{53}^{35},e_{62}^{26},e_{64}^{46},e_{73}^{37},e_{74}^{47}\right\}\end{matrix}\right\},$$

$$\frac{A_0^{\frac12}A_1^{\frac12}U_0}{S_0^-S_1^-}\left\{\begin{matrix}q^{u+\frac12}\left\{\begin{matrix}-\mathbf{1}\left\{e_{32}^{51},e_{42}^{61},e_{43}^{71}\right\}\\ +1\left\{e_{51}^{32},e_{61}^{42},e_{71}^{43}\right\}\end{matrix}\right\}\\ q^{u-\frac12}\left\{\begin{matrix}+\mathbf{1}\left\{e_{23}^{51},e_{24}^{61},e_{34}^{71}\right\}\\ -1\left\{e_{51}^{23},e_{61}^{24},e_{71}^{34}\right\}\end{matrix}\right\}\\ \overline{q}^{u+\frac12}\left\{\begin{matrix}+\mathbf{1}\left\{e_{23}^{15},e_{24}^{16},e_{34}^{17}\right\}\\ -1\left\{e_{15}^{23},e_{16}^{24},e_{17}^{34}\right\}\end{matrix}\right\}\\ \overline{q}^{u-\frac12}\left\{\begin{matrix}-\mathbf{1}\left\{e_{32}^{15},e_{42}^{16},e_{43}^{17}\right\}\\ +1\left\{e_{15}^{32},e_{16}^{42},e_{17}^{43}\right\}\end{matrix}\right\}\end{matrix}\right\}, \qquad \frac{A_1^{\frac12}A_2^{\frac12}U_0S_0^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}q^{u+\frac12}\left\{\begin{matrix}+\mathbf{1}\left\{e_{82}^{65},e_{83}^{75},e_{84}^{76}\right\}\\ -1\left\{e_{65}^{82},e_{75}^{83},e_{76}^{84}\right\}\end{matrix}\right\}\\ q^{u-\frac12}\left\{\begin{matrix}-\mathbf{1}\left\{e_{82}^{56},e_{83}^{57},e_{84}^{67}\right\}\\ +1\left\{e_{56}^{82},e_{57}^{83},e_{67}^{84}\right\}\end{matrix}\right\}\\ \overline{q}^{u+\frac12}\left\{\begin{matrix}-\mathbf{1}\left\{e_{28}^{56},e_{38}^{57},e_{48}^{67}\right\}\\ +1\left\{e_{56}^{28},e_{57}^{38},e_{67}^{48}\right\}\end{matrix}\right\}\\ \overline{q}^{u-\frac12}\left\{\begin{matrix}+\mathbf{1}\left\{e_{28}^{65},e_{38}^{75},e_{48}^{76}\right\}\\ -1\left\{e_{65}^{28},e_{75}^{38},e_{76}^{48}\right\}\end{matrix}\right\}\end{matrix}\right\},$$

$$\frac{U_0}{\Delta^2 S_0^- S_1^- S_2^-}\left\{f_6(q)\left\{\begin{matrix}+\mathbf{1}\left\{e_{45}^{63},-qe_{45}^{72},e_{36}^{72}\right\}\\ -1\left\{e_{63}^{45},-qe_{72}^{45},e_{72}^{36}\right\}\end{matrix}\right\},f_6(\overline{q})\left\{\begin{matrix}+\mathbf{1}\left\{e_{36}^{54},-\overline{q}e_{27}^{54},e_{27}^{63}\right\}\\ -1\left\{e_{54}^{36},-\overline{q}e_{54}^{27},e_{63}^{27}\right\}\end{matrix}\right\}\right\},$$

$$\frac{A_1U_0^2}{S_0^- S_1^- S_2^-}\left\{\begin{matrix}\overline{q}^u\left\{-q\left\{\begin{matrix}e_{45}^{36}\\ e_{36}^{45}\end{matrix}\right\},\left\{\begin{matrix}e_{45}^{27}\\ e_{27}^{45}\end{matrix}\right\},-\overline{q}\left\{\begin{matrix}e_{27}^{36}\\ e_{36}^{27}\end{matrix}\right\}\right\}\\ q^u\left\{-\overline{q}\left\{\begin{matrix}e_{54}^{63}\\ e_{63}^{54}\end{matrix}\right\},\left\{\begin{matrix}e_{54}^{72}\\ e_{72}^{54}\end{matrix}\right\},-q\left\{\begin{matrix}e_{72}^{63}\\ e_{63}^{72}\end{matrix}\right\}\right\}\end{matrix}\right\},$$

$$\frac{A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}U_0U_1}{S_0^- S_1^- S_2^-}\left\{\begin{matrix}\overline{q}^u\left\{q\left\{\begin{matrix}e_{72}^{18}\\ e_{18}^{72}\end{matrix}\right\},-\left\{\begin{matrix}e_{63}^{18}\\ e_{18}^{63}\end{matrix}\right\},\overline{q}\left\{\begin{matrix}e_{54}^{18}\\ e_{18}^{54}\end{matrix}\right\}\right\}\\ q^u\left\{\overline{q}\left\{\begin{matrix}e_{27}^{81}\\ e_{81}^{27}\end{matrix}\right\},-\left\{\begin{matrix}e_{36}^{81}\\ e_{81}^{36}\end{matrix}\right\},q\left\{\begin{matrix}e_{45}^{81}\\ e_{81}^{45}\end{matrix}\right\}\right\}\end{matrix}\right\},$$

$$\frac{A_0^{\frac{1}{2}}A_1A_2^{\frac{1}{2}}U_0}{S_0^- S_1^- S_2^-}\left\{\begin{matrix}\overline{q}^{2u}\left\{\overline{q}\left\{\begin{matrix}+\mathbf{e}_{\mathbf{27}}^{\mathbf{18}}\\ -e_{18}^{27}\end{matrix}\right\},-\left\{\begin{matrix}+\mathbf{e}_{\mathbf{36}}^{\mathbf{18}}\\ -e_{18}^{36}\end{matrix}\right\},q\left\{\begin{matrix}+\mathbf{e}_{\mathbf{45}}^{\mathbf{18}}\\ -e_{18}^{45}\end{matrix}\right\}\right\}\\ q^{2u}\left\{q\left\{\begin{matrix}+\mathbf{e}_{\mathbf{81}}^{\mathbf{72}}\\ -e_{72}^{81}\end{matrix}\right\},-\left\{\begin{matrix}+\mathbf{e}_{\mathbf{81}}^{\mathbf{63}}\\ -e_{63}^{81}\end{matrix}\right\},\overline{q}\left\{\begin{matrix}+\mathbf{e}_{\mathbf{81}}^{\mathbf{54}}\\ -e_{54}^{81}\end{matrix}\right\}\right\}\end{matrix}\right\},$$

where:

$$\begin{aligned}
f_1(q) &= -2\overline{q}+(q^{1+2\alpha}+\overline{q}^{1+2\alpha})-\overline{q}^{2u}(q-\overline{q})\\
f_2(q) &= -2q+(\overline{q}^{3+2\alpha}+q^{3+2\alpha})+q^{2u}(q-\overline{q})\\
f_3(q) &= -\overline{q}^u(2\overline{q}^2-(q^{2+2\alpha}+\overline{q}^{2+2\alpha})+\overline{q}^{2u}(q^2-\overline{q}^2))\\
f_4(q) &= \overline{q}^u(-2+(q^{2+2\alpha}+\overline{q}^{2+2\alpha})-(q^{2-2u}+\overline{q}^{2-2u})+(q^{2u}+\overline{q}^{2u}))\\
f_5(q) &= \overline{q}^u(-2q^2+(q^{2+2\alpha}+\overline{q}^{2+2\alpha})+q^{2u}(q^2-\overline{q}^2))\\
f_6(q) &= q(q+\overline{q})-(q^{2+2\alpha}+\overline{q}^{2+2\alpha})-q^{2u-1}(q-\overline{q}).
\end{aligned}$$

$\check{R}^3$ has 139 nonzero components:

$$1\left\{e_{11}^{11}\right\},\qquad -q^{2\alpha}\left\{e_{22}^{22},e_{33}^{33},e_{44}^{44},\right\},\qquad q^{4\alpha+2}\left\{e_{55}^{55},e_{66}^{66},e_{77}^{77}\right\},\qquad -q^{6\alpha+6}\left\{e_{88}^{88}\right\},$$

$$-\Delta q^{\alpha}A_0\left\{e_{21}^{21},e_{31}^{31},e_{41}^{41}\right\},\qquad -\Delta q^{5\alpha+4}A_2\left\{e_{87}^{87},e_{86}^{86},e_{85}^{85}\right\},$$
$$\Delta q^{2\alpha+1}\left\{e_{32}^{32},e_{42}^{42},e_{43}^{43}\right\},\qquad -\Delta q^{4\alpha+3}\left\{e_{76}^{76},e_{75}^{75},e_{65}^{65}\right\},$$
$$\Delta^2q^{2\alpha+1}A_0A_1\left\{e_{51}^{51},e_{61}^{61},e_{71}^{71}\right\},\qquad -\Delta^2q^{4\alpha+3}A_1A_2\left\{e_{84}^{84},e_{83}^{83},e_{82}^{82}\right\},$$
$$-\Delta^2q^{3\alpha+2}A_1\left\{e_{63}^{63}\right\},\qquad -\Delta q^{3\alpha+3}A_1(q^2-\overline{q}^2)\left\{e_{72}^{72}\right\},$$
$$\Delta q^{3\alpha+1}A_1\left\{e_{52}^{52},e_{53}^{53},e_{62}^{62},e_{64}^{64},e_{73}^{73},e_{74}^{74}\right\},\qquad -\Delta^3q^{3\alpha+3}A_0A_1A_2\left\{e_{81}^{81}\right\},$$

$$q^{\alpha}\left\{\begin{matrix}-\mathbf{1}\left\{e_{21}^{12},e_{31}^{13},e_{41}^{14}\right\}\\ +1\left\{e_{12}^{21},e_{13}^{31},e_{14}^{41}\right\}\end{matrix}\right\},\qquad q^{5\alpha+4}\left\{\begin{matrix}-\mathbf{1}\left\{e_{87}^{78},e_{86}^{68},e_{85}^{58}\right\}\\ +1\left\{e_{78}^{87},e_{68}^{86},e_{58}^{85}\right\}\end{matrix}\right\},$$
$$q^{2\alpha}\left\{\begin{matrix}e_{51}^{15},e_{61}^{16},e_{71}^{17}\\ e_{15}^{51},e_{16}^{61},e_{17}^{71}\end{matrix}\right\},\qquad -q^{4\alpha+2}\left\{\begin{matrix}e_{48}^{84},e_{38}^{83},e_{28}^{82}\\ e_{84}^{48},e_{83}^{38},e_{82}^{28}\end{matrix}\right\},$$
$$-q^{2\alpha+1}\left\{\begin{matrix}e_{32}^{23},e_{42}^{24},e_{43}^{34}\\ e_{23}^{32},e_{24}^{42},e_{34}^{43}\end{matrix}\right\},\qquad q^{4\alpha+3}\left\{\begin{matrix}e_{67}^{76},e_{57}^{75},e_{56}^{65}\\ e_{76}^{67},e_{75}^{57},e_{65}^{56}\end{matrix}\right\},$$
$$q^{3\alpha+1}\left\{\begin{matrix}-\mathbf{1}\left\{e_{25}^{52},e_{26}^{62},e_{35}^{53},e_{37}^{73},e_{46}^{64},e_{47}^{74}\right\}\\ +1\left\{e_{52}^{25},e_{62}^{26},e_{53}^{35},e_{73}^{37},e_{64}^{46},e_{74}^{47}\right\}\end{matrix}\right\},\quad q^{3\alpha+2}\left\{\begin{matrix}-\mathbf{1}\left\{e_{27}^{72},e_{36}^{63},e_{45}^{54}\right\}\\ +1\left\{e_{72}^{27},e_{63}^{36},e_{54}^{45}\right\}\end{matrix}\right\},\quad q^{3\alpha}\left\{\begin{matrix}-\mathbf{1}\left\{e_{81}^{18}\right\}\\ +1\left\{e_{18}^{81}\right\}\end{matrix}\right\},$$

$$\Delta q^{2\alpha+1}A_0A_1\left\{\overline{q}^{\frac{1}{2}}\left\{\begin{matrix}+\mathbf{1}\left\{e_{23}^{51},e_{24}^{61},e_{34}^{71}\right\}\\ -1\left\{e_{51}^{23},e_{61}^{24},e_{71}^{34}\right\}\end{matrix}\right\},q^{\frac{1}{2}}\left\{\begin{matrix}-\mathbf{1}\left\{e_{32}^{51},e_{42}^{61},e_{43}^{71}\right\}\\ +1\left\{e_{51}^{32},e_{61}^{42},e_{71}^{43}\right\}\end{matrix}\right\}\right\},$$

$$\Delta q^{4\alpha+3} A_1 A_2 \left\{ \overline{q}^{\frac{1}{2}} \left\{ \begin{array}{l} +\mathbf{1} \left\{ e^{56}_{82}, e^{57}_{83}, e^{67}_{84} \right\} \\ -1 \left\{ e^{82}_{56}, e^{83}_{57}, e^{84}_{67} \right\} \end{array} \right\}, q^{\frac{1}{2}} \left\{ \begin{array}{l} -\mathbf{1} \left\{ e^{65}_{82}, e^{75}_{83}, e^{76}_{84} \right\} \\ +1 \left\{ e^{82}_{65}, e^{83}_{75}, e^{84}_{76} \right\} \end{array} \right\} \right\},$$

$$\Delta q^{3\alpha+\frac{5}{2}} \left\{ \overline{q}^{\frac{1}{2}} \left\{ \begin{array}{l} +\mathbf{1} \left\{ e^{63}_{45}, e^{72}_{36} \right\} \\ -1 \left\{ e^{45}_{63}, e^{36}_{72} \right\} \end{array} \right\}, q^{\frac{1}{2}} \left\{ \begin{array}{l} -\mathbf{1} \left\{ e^{72}_{45} \right\} \\ +1 \left\{ e^{45}_{72} \right\} \end{array} \right\} \right\},$$

$$\Delta q^{3\alpha+3} A_1 \left\{ \overline{q} \left\{ \begin{array}{c} e^{63}_{54} \\ e^{54}_{63}, \end{array} \right\}, - \left\{ \begin{array}{c} e^{72}_{54} \\ e^{54}_{72} \end{array} \right\}, q \left\{ \begin{array}{c} e^{63}_{72} \\ e^{72}_{63} \end{array} \right\} \right\},$$

$$\Delta q^{3\alpha+2} A_0 A_2 \left\{ -\overline{q} \left\{ \begin{array}{c} e^{81}_{27} \\ e^{27}_{81} \end{array} \right\}, \left\{ \begin{array}{c} e^{81}_{36} \\ e^{36}_{81} \end{array} \right\} - q \left\{ \begin{array}{c} e^{81}_{45} \\ e^{45}_{81} \end{array} \right\} \right\},$$

$$\Delta^2 q^{3\alpha+3} A_0 A_1 A_2 \left\{ \overline{q} \left\{ \begin{array}{c} -\mathbf{e^{54}_{81}} \\ +e^{81}_{54} \end{array} \right\}, - \left\{ \begin{array}{c} -\mathbf{e^{63}_{81}} \\ +e^{81}_{63} \end{array} \right\}, q \left\{ \begin{array}{c} -\mathbf{e^{72}_{81}} \\ +e^{81}_{72} \end{array} \right\} \right\}.$$

R matrices for m = 4

Here, $[i]$ is 0 for $i \in \{1; 6, 7, 8, 9, 10, 11; 16\}$, and 1 for $i \in \{2, 3, 4, 5; 12, 13, 14, 15\}$, $\check{R}^4(u)$ has 1296 nonzero components:

$$1 \left\{ e^{1,1}_{1,1} \right\}, \qquad \frac{S_0^+ S_1^+}{S_0^- S_1^-} \left\{ e^{6,6}_{6,6}, e^{7,7}_{7,7}, e^{8,8}_{8,8}, e^{9,9}_{9,9}, e^{10,10}_{10,10}, e^{11,11}_{11,11} \right\}, \qquad \frac{S_0^+ S_1^+ S_2^+ S_3^+}{S_0^- S_1^- S_2^- S_3^-} \left\{ e^{16,16}_{16,16} \right\},$$

$$\frac{S_0^+}{S_0^-} \left\{ e^{2,2}_{2,2}, e^{3,3}_{3,3}, e^{4,4}_{4,4}, e^{5,5}_{5,5} \right\}, \qquad \frac{S_0^+ S_1^+ S_2^+}{S_0^- S_1^- S_2^-} \left\{ e^{12,12}_{12,12}, e^{13,13}_{13,13}, e^{14,14}_{14,14}, e^{15,15}_{15,15} \right\},$$

$$\frac{A_0}{S_0^-} \left\{ \begin{array}{l} \overline{q}^u \left\{ e^{1,2}_{1,2}, e^{1,3}_{1,3}, e^{1,4}_{1,4}, e^{1,5}_{1,5} \right\} \\ q^u \left\{ e^{2,1}_{2,1}, e^{3,1}_{3,1}, e^{4,1}_{4,1}, e^{5,1}_{5,1} \right\} \end{array} \right\}, \qquad \frac{A_3 S_0^+ S_1^+ S_2^+}{S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{l} q^u \left\{ e^{16,15}_{16,15}, e^{16,14}_{16,14}, e^{16,13}_{16,13}, e^{16,12}_{16,12} \right\} \\ \overline{q}^u \left\{ e^{15,16}_{15,16}, e^{14,16}_{14,16}, e^{13,16}_{13,16}, e^{12,16}_{12,16} \right\} \end{array} \right\},$$

$$\frac{A_0 A_1}{S_0^- S_1^-} \left\{ \begin{array}{l} \overline{q}^{2u} \left\{ e^{1,6}_{1,6}, e^{1,7}_{1,7}, e^{1,8}_{1,8}, e^{1,9}_{1,9}, e^{1,10}_{1,10}, e^{1,11}_{1,11} \right\} \\ q^{2u} \left\{ e^{6,1}_{6,1}, e^{7,1}_{7,1}, e^{8,1}_{8,1}, e^{9,1}_{9,1}, e^{10,1}_{10,1}, e^{11,1}_{11,1} \right\} \end{array} \right\}, \qquad \frac{A_0 A_1 A_2}{S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} \overline{q}^{3u} \left\{ e^{1,12}_{1,12}, e^{1,13}_{1,13}, e^{1,14}_{1,14}, e^{1,15}_{1,15} \right\} \\ q^{3u} \left\{ e^{12,1}_{12,1}, e^{13,1}_{13,1}, e^{14,1}_{14,1}, e^{15,1}_{15,1} \right\} \end{array} \right\},$$

$$\frac{A_2 A_3 S_0^+ S_1^+}{S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{l} q^{2u} \left\{ e^{16,11}_{16,11}, e^{16,10}_{16,10}, e^{16,9}_{16,9}, e^{16,8}_{16,8}, e^{16,7}_{16,7}, e^{16,6}_{16,6} \right\} \\ \overline{q}^{2u} \left\{ e^{11,16}_{11,16}, e^{10,16}_{10,16}, e^{9,16}_{9,16}, e^{8,16}_{8,16}, e^{7,16}_{7,16}, e^{6,16}_{6,16} \right\} \end{array} \right\},$$

$$\frac{A_1 A_2 A_3 S_0^+}{S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{l} q^{3u} \left\{ e^{16,5}_{16,5}, e^{16,4}_{16,4}, e^{16,3}_{16,3}, e^{16,2}_{16,2} \right\} \\ \overline{q}^{3u} \left\{ e^{5,16}_{5,16}, e^{4,16}_{4,16}, e^{3,16}_{3,16}, e^{2,16}_{2,16} \right\} \end{array} \right\}, \qquad \frac{A_0 A_1 A_2 A_3}{S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{l} \overline{q}^{4u} e^{1,16}_{1,16}, \\ q^{4u} e^{16,1}_{16,1} \end{array} \right\},$$

$$\frac{A_1 S_0^+}{S_0^- S_1^-} \left\{ \begin{array}{l} \overline{q}^u \left\{ e^{2,6}_{2,6}, e^{2,7}_{2,7}, e^{2,8}_{2,8}, e^{3,6}_{3,6}, e^{3,9}_{3,9}, e^{3,10}_{3,10}, e^{4,7}_{4,7}, e^{4,9}_{4,9}, e^{4,11}_{4,11}, e^{5,8}_{5,8}, e^{5,10}_{5,10}, e^{5,11}_{5,11} \right\} \\ q^u \left\{ e^{6,2}_{6,2}, e^{7,2}_{7,2}, e^{8,2}_{8,2}, e^{6,3}_{6,3}, e^{9,3}_{9,3}, e^{10,3}_{10,3}, e^{7,4}_{7,4}, e^{9,4}_{9,4}, e^{11,4}_{11,4}, e^{8,5}_{8,5}, e^{10,5}_{10,5}, e^{11,5}_{11,5} \right\} \end{array} \right\},$$

$$\frac{A_2 S_0^+ S_1^+}{S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} q^u \left\{ e^{15,11}_{15,11}, e^{15,10}_{15,10}, e^{15,9}_{15,9}, e^{14,11}_{14,11}, e^{14,8}_{14,8}, e^{14,7}_{14,7}, e^{13,10}_{13,10}, e^{13,8}_{13,8}, e^{13,6}_{13,6}, e^{12,9}_{12,9}, e^{12,7}_{12,7}, e^{12,6}_{12,6} \right\} \\ \overline{q}^u \left\{ e^{11,15}_{11,15}, e^{10,15}_{10,15}, e^{9,15}_{9,15}, e^{11,14}_{11,14}, e^{8,14}_{8,14}, e^{7,14}_{7,14}, e^{10,13}_{10,13}, e^{8,13}_{8,13}, e^{6,13}_{6,13}, e^{9,12}_{9,12}, e^{7,12}_{7,12}, e^{6,12}_{6,12} \right\} \end{array} \right\},$$

$$\frac{A_1 A_2 S_0^+}{S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} \overline{q}^{2u} \left\{ e^{2,12}_{2,12}, e^{2,13}_{2,13}, e^{2,14}_{2,14}, e^{3,12}_{3,12}, e^{3,13}_{3,13}, e^{3,15}_{3,15}, e^{4,12}_{4,12}, e^{4,14}_{4,14}, e^{4,15}_{4,15}, e^{5,13}_{5,13}, e^{5,14}_{5,14}, e^{5,15}_{5,15} \right\} \\ q^{2u} \left\{ e^{12,2}_{12,2}, e^{13,2}_{13,2}, e^{14,2}_{14,2}, e^{12,3}_{12,3}, e^{13,3}_{13,3}, e^{15,3}_{15,3}, e^{12,4}_{12,4}, e^{14,4}_{14,4}, e^{15,4}_{15,4}, e^{13,5}_{13,5}, e^{14,5}_{14,5}, e^{15,5}_{15,5} \right\} \end{array} \right\},$$

$$\frac{1}{\Delta^2 S_0^- S_1^-} \left\{ \begin{array}{l} g_1(q) \left\{ e^{3,2}_{3,2}, e^{4,2}_{4,2}, e^{5,2}_{5,2}, e^{4,3}_{4,3}, e^{5,3}_{5,3}, e^{5,4}_{5,4} \right\} \\ g_1(\overline{q}) \left\{ e^{2,3}_{2,3}, e^{2,4}_{2,4}, e^{2,5}_{2,5}, e^{3,4}_{3,4}, e^{3,5}_{3,5}, e^{4,5}_{4,5} \right\} \end{array} \right\}, \qquad \frac{A_1}{\Delta^2 S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} g_2(q) \left\{ e^{3,7}_{3,7}, e^{3,8}_{3,8}, e^{4,8}_{4,8}, e^{4,10}_{4,10} \right\} \\ g_2(\overline{q}) \left\{ e^{7,3}_{7,3}, e^{8,3}_{8,3}, e^{8,4}_{8,4}, e^{10,4}_{10,4} \right\} \end{array} \right\},$$

$$\frac{A_1}{\Delta^2 S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} \overline{q}^u g_3(q) \left\{ e^{9,2}_{9,2}, e^{10,2}_{10,2}, e^{11,2}_{11,2}, e^{11,3}_{11,3} \right\}, q^u g_3(q) \left\{ e^{4,6}_{4,6}, e^{5,6}_{5,6}, e^{5,7}_{5,7}, e^{5,9}_{5,9} \right\}, \\ q^u g_3(\overline{q}) \left\{ e^{2,9}_{2,9}, e^{2,10}_{2,10}, e^{2,11}_{2,11}, e^{3,11}_{3,11} \right\}, \overline{q}^u g_3(\overline{q}) \left\{ e^{6,4}_{6,4}, e^{6,5}_{6,5}, e^{7,5}_{7,5}, e^{9,5}_{9,5} \right\} \end{array} \right\},$$

$$\frac{S_0^+}{\Delta^2 S_0^- S_1^- S_2^-} \left\{ \begin{array}{l} g_4(q) \left\{ e^{7,6}_{7,6}, e^{8,6}_{8,6}, e^{9,6}_{9,6}, e^{10,6}_{10,6}, e^{8,7}_{8,7}, e^{9,7}_{9,7}, e^{11,7}_{11,7}, e^{10,8}_{10,8}, e^{11,8}_{11,8}, e^{10,9}_{10,9}, e^{11,9}_{11,9}, e^{11,10}_{11,10} \right\} \\ g_4(\overline{q}) \left\{ e^{6,7}_{6,7}, e^{6,8}_{6,8}, e^{6,9}_{6,9}, e^{6,10}_{6,10}, e^{7,8}_{7,8}, e^{7,9}_{7,9}, e^{7,11}_{7,11}, e^{8,10}_{8,10}, e^{8,11}_{8,11}, e^{9,10}_{9,10}, e^{9,11}_{9,11}, e^{10,11}_{10,11} \right\} \end{array} \right\},$$

$$\frac{A_2 S_0^+}{\Delta^2 S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{c} \overline{q}^u g_5(q) \left\{ e^{15,8}_{15,8}, e^{15,7}_{15,7}, e^{15,6}_{15,6}, e^{14,6}_{14,6} \right\}, q^u g_5(q) \left\{ e^{8,12}_{8,12}, e^{10,12}_{10,12}, e^{11,12}_{11,12}, e^{11,13}_{11,13} \right\} \\ q^u g_5(\overline{q}) \left\{ e^{8,15}_{8,15}, e^{7,15}_{7,15}, e^{6,15}_{6,15}, e^{6,14}_{6,14} \right\}, \overline{q}^u g_5(\overline{q}) \left\{ e^{12,8}_{12,8}, e^{12,10}_{12,10}, e^{12,11}_{12,11}, e^{13,11}_{13,11} \right\} \\ q^{2u} g_6(q) \left\{ e^{15,2}_{15,2} \right\}, \overline{q}^{2u} g_6(q) \left\{ e^{5,12}_{5,12} \right\}, q^{2u} g_7(q) \left\{ e^{14,3}_{14,3} \right\}, \overline{q}^{2u} g_7(q) \left\{ e^{4,13}_{4,13} \right\} \\ \overline{q}^{2u} g_6(\overline{q}) \left\{ e^{2,15}_{2,15} \right\}, q^{2u} g_6(\overline{q}) \left\{ e^{12,5}_{12,5} \right\}, \overline{q}^{2u} g_7(\overline{q}) \left\{ e^{3,14}_{3,14} \right\}, q^{2u} g_7(\overline{q}) \left\{ e^{13,4}_{13,4} \right\} \\ g_8(q) \left\{ e^{13,7}_{13,7}, e^{13,9}_{13,9}, e^{14,9}_{14,9}, e^{14,10}_{14,10} \right\}, g_8(\overline{q}) \left\{ e^{7,13}_{7,13}, e^{9,13}_{9,13}, e^{9,14}_{9,14}, e^{10,14}_{10,14} \right\} \end{array} \right\},$$

$$\frac{S_0^+ S_1^+}{\Delta^2 S_0^- S_1^- S_2^- S_3^-} \left\{ \begin{array}{l} g_9(q) \left\{ e^{15,14}_{15,14}, e^{15,13}_{15,13}, e^{15,12}_{15,12}, e^{14,13}_{14,13}, e^{14,12}_{14,12}, e^{13,12}_{13,12} \right\} \\ g_9(\overline{q}) \left\{ e^{14,15}_{14,15}, e^{13,15}_{13,15}, e^{13,14}_{13,14}, e^{12,15}_{12,15}, e^{12,14}_{12,14}, e^{12,13}_{12,13} \right\} \end{array} \right\},$$

$$\frac{1}{\Delta^4 S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} g_{10}(\overline{q})\left\{e^{6,11}_{6,11}\right\},\, g_{11}(\overline{q})\left\{e^{7,10}_{7,10}\right\}\, g_{12}(\overline{q})\left\{e^{8,9}_{8,9}\right\} \\ g_{10}(q)\left\{e^{11,6}_{11,6}\right\},\, g_{11}(q)\left\{e^{10,7}_{10,7}\right\},\, g_{12}(q)\left\{e^{9,8}_{9,8}\right\}\end{array}\right\},$$

$$\frac{U_0}{S_0^-}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{1,2}_{2,1}, e^{1,3}_{3,1}, e^{1,4}_{4,1}, e^{1,5}_{5,1}\right\} \\ -1\left\{e^{2,1}_{1,2}, e^{3,1}_{1,3}, e^{4,1}_{1,4}, e^{5,1}_{1,5}\right\}\end{array}\right\}, \qquad \frac{U_0 S_0^+ S_1^+ S_2^+}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,15}_{15,16}, e^{16,14}_{14,16}, e^{16,13}_{13,16}, e^{16,12}_{12,16}\right\} \\ +1\left\{e^{15,16}_{16,15}, e^{14,16}_{16,14}, e^{13,16}_{16,13}, e^{12,16}_{16,12}\right\}\end{array}\right\},$$

$$\frac{U_0 U_1}{S_0^- S_1^-}\left\{\begin{array}{l} e^{1,6}_{6,1}, e^{1,7}_{7,1}, e^{1,8}_{8,1}, e^{1,9}_{9,1}, e^{1,10}_{10,1}, e^{1,11}_{11,1} \\ e^{6,1}_{1,6}, e^{7,1}_{1,7}, e^{8,1}_{1,8}, e^{9,1}_{1,9}, e^{10,1}_{1,10}, e^{11,1}_{1,11}\end{array}\right\}, \qquad \frac{U_0 U_1 S_0^+ S_1^+}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} e^{16,11}_{11,16}, e^{16,10}_{10,16}, e^{16,9}_{9,16}, e^{16,8}_{8,16}, e^{16,7}_{7,16}, e^{16,6}_{6,16} \\ e^{11,16}_{16,11}, e^{10,16}_{16,10}, e^{9,16}_{16,9}, e^{8,16}_{16,8}, e^{7,16}_{16,7}, e^{6,16}_{16,6}\end{array}\right\},$$

$$\frac{U_0 U_1 U_1}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{1,12}_{12,1}, e^{1,13}_{13,1}, e^{1,14}_{14,1}, e^{1,15}_{15,1}\right\} \\ -1\left\{e^{12,1}_{1,12}, e^{13,1}_{1,13}, e^{14,1}_{1,14}, e^{15,1}_{1,15}\right\}\end{array}\right\}, \qquad \frac{U_0 U_1 U_1 S_0^+}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,5}_{5,16}, e^{16,4}_{4,16}, e^{16,3}_{3,16}, e^{16,2}_{2,16}\right\} \\ +1\left\{e^{5,16}_{16,5}, e^{4,16}_{16,4}, e^{3,16}_{16,3}, e^{2,16}_{16,2}\right\}\end{array}\right\},$$

$$-\frac{{U_0}^2}{S_0^- S_1^-}\left\{\begin{array}{l} e^{2,3}_{3,2}, e^{2,4}_{4,2}, e^{2,5}_{5,2}, e^{3,4}_{4,3}, e^{3,5}_{5,3}, e^{4,5}_{5,4} \\ e^{3,2}_{2,3}, e^{4,2}_{2,4}, e^{5,2}_{2,5}, e^{4,3}_{3,4}, e^{5,3}_{3,5}, e^{5,4}_{4,5}\end{array}\right\}, \qquad \frac{{U_0}^2 S_0^+ S_1^+}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} e^{15,14}_{14,15}, e^{15,13}_{13,15}, e^{15,12}_{12,15}, e^{14,13}_{13,14}, e^{14,12}_{12,14}, e^{13,12}_{12,13} \\ e^{14,15}_{15,14}, e^{13,15}_{15,13}, e^{12,15}_{15,12}, e^{13,14}_{14,13}, e^{12,14}_{14,12}, e^{12,13}_{13,12}\end{array}\right\},$$

$$\frac{U_0^2 U_1}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{6,4}_{4,6}, e^{6,5}_{5,6}, e^{7,3}_{3,7}, e^{7,5}_{5,7}, e^{8,3}_{3,8}, e^{8,4}_{4,8}, e^{9,2}_{2,9}, e^{9,5}_{5,9}, e^{10,2}_{2,10}, e^{10,4}_{4,10}, e^{11,2}_{2,11}, e^{11,3}_{3,11}\right\} \\ -1\left\{e^{4,6}_{6,4}, e^{5,6}_{6,5}, e^{3,7}_{7,3}, e^{5,7}_{7,5}, e^{3,8}_{8,3}, e^{4,8}_{8,4}, e^{2,9}_{9,2}, e^{5,9}_{9,5}, e^{2,10}_{10,2}, e^{4,10}_{10,4}, e^{2,11}_{11,2}, e^{3,11}_{11,3}\right\}\end{array}\right\},$$

$$-\frac{{U_0}^2 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} e^{6,7}_{7,6}, e^{6,8}_{8,6}, e^{6,9}_{9,6}, e^{6,10}_{10,6}, e^{7,8}_{8,7}, e^{7,9}_{9,7}, e^{7,11}_{11,7}, e^{8,10}_{10,8}, e^{8,11}_{11,8}, e^{9,10}_{10,9}, e^{9,11}_{11,9}, e^{11,10}_{10,11} \\ e^{7,6}_{6,7}, e^{8,6}_{6,8}, e^{9,6}_{6,9}, e^{10,6}_{6,10}, e^{8,7}_{7,8}, e^{9,7}_{7,9}, e^{11,7}_{7,11}, e^{10,8}_{8,10}, e^{11,8}_{8,11}, e^{11,9}_{9,11}, e^{10,9}_{9,10}, e^{10,11}_{11,10}\end{array}\right\},$$

$$\frac{U_0^2 U_1 S_0^+}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,14}_{14,6}, e^{6,15}_{15,6}, e^{7,13}_{13,7}, e^{7,15}_{15,7}, e^{8,12}_{12,8}, e^{8,15}_{15,8}, e^{9,13}_{13,9}, e^{9,14}_{14,9}, e^{10,12}_{12,10}, e^{10,14}_{14,10}, e^{11,12}_{12,11}, e^{11,13}_{13,11}\right\} \\ +1\left\{e^{14,6}_{6,14}, e^{15,6}_{6,15}, e^{13,7}_{7,13}, e^{15,7}_{7,15}, e^{12,8}_{8,12}, e^{15,8}_{8,15}, e^{13,9}_{9,13}, e^{14,9}_{9,14}, e^{12,10}_{10,12}, e^{14,10}_{10,14}, e^{12,11}_{11,12}, e^{13,11}_{11,13}\right\}\end{array}\right\},$$

$$\frac{U_0 S_0^+ S_1^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{6,12}_{12,6}, e^{6,13}_{13,6}, e^{7,12}_{12,7}, e^{7,14}_{14,7}, e^{8,13}_{13,8}, e^{8,14}_{14,8}, e^{9,12}_{12,9}, e^{9,15}_{15,9}, e^{10,13}_{13,10}, e^{10,15}_{15,10}, e^{11,14}_{14,11}, e^{11,15}_{15,11}\right\} \\ -1\left\{e^{12,6}_{6,12}, e^{13,6}_{6,13}, e^{12,7}_{7,12}, e^{14,7}_{7,14}, e^{13,8}_{8,13}, e^{14,8}_{8,14}, e^{12,9}_{9,12}, e^{15,9}_{9,15}, e^{13,10}_{10,13}, e^{15,10}_{10,15}, e^{14,11}_{11,14}, e^{15,11}_{11,15}\right\}\end{array}\right\},$$

$$\frac{U_0 S_0^+}{S_0^- S_1^-}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,2}_{2,6}, e^{7,2}_{2,7}, e^{8,2}_{2,8}, e^{6,3}_{3,6}, e^{9,3}_{3,9}, e^{10,3}_{3,10}, e^{7,4}_{4,7}, e^{9,4}_{4,9}, e^{11,4}_{4,11}, e^{8,5}_{5,8}, e^{10,5}_{5,10}, e^{11,5}_{5,11}\right\} \\ +1\left\{e^{2,6}_{6,2}, e^{2,7}_{7,2}, e^{2,8}_{8,2}, e^{3,6}_{6,3}, e^{3,9}_{9,3}, e^{3,10}_{10,3}, e^{4,7}_{7,4}, e^{4,9}_{9,4}, e^{4,11}_{11,4}, e^{5,8}_{8,5}, e^{5,10}_{10,5}, e^{5,11}_{11,5}\right\}\end{array}\right\},$$

$$\frac{U_0 U_1 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} e^{2,12}_{12,2}, e^{2,13}_{13,2}, e^{2,14}_{14,2}, e^{3,12}_{12,3}, e^{3,13}_{13,3}, e^{3,15}_{15,3}, e^{4,12}_{12,4}, e^{4,14}_{14,4}, e^{4,15}_{15,4}, e^{5,13}_{13,5}, e^{5,14}_{14,5}, e^{5,15}_{15,5} \\ e^{12,2}_{2,12}, e^{13,2}_{2,13}, e^{14,2}_{2,14}, e^{12,3}_{3,12}, e^{13,3}_{3,13}, e^{15,3}_{3,15}, e^{12,4}_{4,12}, e^{14,4}_{4,14}, e^{15,4}_{4,15}, e^{13,5}_{5,13}, e^{14,5}_{5,14}, e^{15,5}_{5,15}\end{array}\right\},$$

$$\frac{U_0^2 U_1^2}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} e^{6,11}_{11,6}, e^{7,10}_{10,7}, e^{8,9}_{9,8} \\ e^{9,8}_{8,9}, e^{10,7}_{7,10}, e^{11,6}_{6,11}\end{array}\right\}, \qquad -\frac{U_0^2 U_1 U_1}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} e^{2,15}_{15,2}, e^{3,14}_{14,3}, e^{4,13}_{13,4}, e^{5,12}_{12,5} \\ e^{15,2}_{2,15}, e^{14,3}_{3,14}, e^{13,4}_{4,13}, e^{12,5}_{5,12}\end{array}\right\}, \qquad \frac{U_0 U_1 U_1 U_1}{S_0^- S_1^- S_2^- S_3^-}\left\{\begin{array}{l} e^{1,16}_{16,1} \\ e^{16,1}_{1,16}\end{array}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_1^{\frac{1}{2}} U_0}{S_0^- S_1^-}\left\{\begin{array}{l} -\mathbf{q}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{6,1}_{3,2}, e^{7,1}_{4,2}, e^{8,1}_{5,2}, e^{9,1}_{4,3}, e^{10,1}_{5,3}, e^{11,1}_{5,4}\right\},\, +\overline{\mathbf{q}}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{1,6}_{2,3}, e^{1,7}_{2,4}, e^{1,8}_{2,5}, e^{1,9}_{3,4}, e^{1,10}_{3,5}, e^{1,11}_{4,5}\right\} \\ +q^{u+\frac{1}{2}}\left\{e^{3,2}_{6,1}, e^{4,2}_{7,1}, e^{5,2}_{8,1}, e^{4,3}_{9,1}, e^{5,3}_{10,1}, e^{5,4}_{11,1}\right\},\, -\overline{q}^{u+\frac{1}{2}}\left\{e^{2,3}_{1,6}, e^{2,4}_{1,7}, e^{2,5}_{1,8}, e^{3,4}_{1,9}, e^{3,5}_{1,10}, e^{4,5}_{1,11}\right\} \\ +\mathbf{q}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{6,1}_{2,3}, e^{7,1}_{2,4}, e^{8,1}_{2,5}, e^{9,1}_{3,4}, e^{10,1}_{3,5}, e^{11,1}_{4,5}\right\},\, -\overline{\mathbf{q}}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{1,6}_{3,2}, e^{1,7}_{4,2}, e^{1,8}_{5,2}, e^{1,9}_{4,3}, e^{1,10}_{5,3}, e^{1,11}_{5,4}\right\} \\ -q^{u-\frac{1}{2}}\left\{e^{2,3}_{6,1}, e^{2,4}_{7,1}, e^{2,5}_{8,1}, e^{3,4}_{9,1}, e^{3,5}_{10,1}, e^{4,5}_{11,1}\right\},\, +\overline{q}^{u-\frac{1}{2}}\left\{e^{3,2}_{1,6}, e^{4,2}_{1,7}, e^{5,2}_{1,8}, e^{4,3}_{1,9}, e^{5,3}_{1,10}, e^{5,4}_{1,11}\right\}\end{array}\right\},$$

$$\frac{A_1 {U_0}^2}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} -q^{u-1}\left\{\begin{array}{l} e^{7,3}_{6,4}, e^{8,3}_{6,5}, e^{8,4}_{7,5}, e^{10,4}_{9,5} \\ e^{6,4}_{7,3}, e^{6,5}_{8,3}, e^{7,5}_{8,4}, e^{9,5}_{10,4}\end{array}\right\},\, q^{u}\left\{\begin{array}{l} e^{9,2}_{6,4}, e^{10,2}_{6,5}, e^{11,2}_{7,5}, e^{11,3}_{9,5} \\ e^{6,4}_{9,2}, e^{6,5}_{10,2}, e^{7,5}_{11,2}, e^{9,5}_{11,3}\end{array}\right\},\, -q^{u+1}\left\{\begin{array}{l} e^{9,2}_{7,3}, e^{10,2}_{8,3}, e^{11,2}_{8,4}, e^{11,3}_{10,4} \\ e^{7,3}_{9,2}, e^{8,3}_{10,2}, e^{8,4}_{11,2}, e^{10,4}_{11,3}\end{array}\right\} \\ -\overline{q}^{u-1}\left\{\begin{array}{l} e^{3,7}_{4,6}, e^{3,8}_{5,6}, e^{4,8}_{5,7}, e^{4,10}_{5,9} \\ e^{4,6}_{3,7}, e^{5,6}_{3,8}, e^{5,7}_{4,8}, e^{5,9}_{4,10}\end{array}\right\},\, \overline{q}^{u}\left\{\begin{array}{l} e^{2,9}_{4,6}, e^{2,10}_{5,6}, e^{2,11}_{5,7}, e^{3,11}_{5,9} \\ e^{4,6}_{2,9}, e^{5,6}_{2,10}, e^{5,7}_{2,11}, e^{5,9}_{3,11}\end{array}\right\},\, -\overline{q}^{u+1}\left\{\begin{array}{l} e^{2,9}_{3,7}, e^{2,10}_{3,8}, e^{2,11}_{4,8}, e^{3,11}_{4,10} \\ e^{3,7}_{2,9}, e^{3,8}_{2,10}, e^{4,8}_{2,11}, e^{4,10}_{3,11}\end{array}\right\}\end{array}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_1 A_2^{\frac{1}{2}} U_0}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} +\mathbf{q}^{\mathbf{2u+1}}\left\{e^{9,2}_{12,1}, e^{10,2}_{13,1}, e^{11,2}_{14,1}, e^{11,3}_{15,1}\right\},\, -\mathbf{q}^{\mathbf{2u}}\left\{e^{7,3}_{12,1}, e^{8,3}_{13,1}, e^{8,4}_{14,1}, e^{10,4}_{15,1}\right\},\, +\mathbf{q}^{\mathbf{2u-1}}\left\{e^{6,4}_{12,1}, e^{6,5}_{13,1}, e^{7,5}_{14,1}, e^{9,5}_{15,1}\right\} \\ -q^{2u+1}\left\{e^{12,1}_{9,2}, e^{13,1}_{10,2}, e^{14,1}_{11,2}, e^{15,1}_{11,3}\right\},\, +q^{2u}\left\{e^{12,1}_{7,3}, e^{13,1}_{8,3}, e^{14,1}_{8,4}, e^{15,1}_{10,4}\right\},\, -q^{2u-1}\left\{e^{12,1}_{6,4}, e^{13,1}_{6,5}, e^{14,1}_{7,5}, e^{15,1}_{9,5}\right\} \\ +\overline{\mathbf{q}}^{\mathbf{2u+1}}\left\{e^{1,12}_{2,9}, e^{1,13}_{2,10}, e^{1,14}_{2,11}, e^{1,15}_{3,11}\right\},\, -\overline{\mathbf{q}}^{\mathbf{2u}}\left\{e^{1,12}_{3,7}, e^{1,13}_{3,8}, e^{1,14}_{4,8}, e^{1,15}_{4,10}\right\},\, +\overline{\mathbf{q}}^{\mathbf{2u-1}}\left\{e^{1,12}_{4,6}, e^{1,13}_{5,6}, e^{1,14}_{5,7}, e^{1,15}_{5,9}\right\} \\ -\overline{q}^{2u+1}\left\{e^{2,9}_{1,12}, e^{2,10}_{1,13}, e^{2,11}_{1,14}, e^{3,11}_{1,15}\right\},\, +\overline{q}^{2u}\left\{e^{3,7}_{1,12}, e^{3,8}_{1,13}, e^{4,8}_{1,14}, e^{4,10}_{1,15}\right\},\, -\overline{q}^{2u-1}\left\{e^{4,6}_{1,12}, e^{5,6}_{1,13}, e^{5,7}_{1,14}, e^{5,9}_{1,15}\right\}\end{array}\right\},$$

$$\frac{A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} U_0 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l} +\mathbf{q}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{7,6}_{12,2}, e^{8,6}_{13,2}, e^{8,7}_{14,2}, e^{9,6}_{12,3}, e^{9,7}_{12,4}, e^{10,6}_{13,3}, e^{10,8}_{13,5}, e^{10,9}_{15,3}, e^{11,7}_{14,4}, e^{11,8}_{14,5}, e^{11,9}_{15,4}, e^{11,10}_{15,5}\right\} \\ -q^{u+\frac{1}{2}}\left\{e^{12,2}_{7,6}, e^{13,2}_{8,6}, e^{14,2}_{8,7}, e^{12,3}_{9,6}, e^{12,4}_{9,7}, e^{13,3}_{10,6}, e^{13,5}_{10,8}, e^{15,3}_{10,9}, e^{14,4}_{11,7}, e^{14,5}_{11,8}, e^{15,4}_{11,9}, e^{15,5}_{11,10}\right\} \\ -\overline{\mathbf{q}}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{6,7}_{2,12}, e^{6,8}_{2,13}, e^{7,8}_{2,14}, e^{6,9}_{3,12}, e^{7,9}_{4,12}, e^{6,10}_{3,13}, e^{8,10}_{5,13}, e^{9,10}_{3,15}, e^{7,11}_{4,14}, e^{8,11}_{5,14}, e^{9,11}_{4,15}, e^{10,11}_{5,15}\right\} \\ +\overline{q}^{u+\frac{1}{2}}\left\{e^{2,12}_{6,7}, e^{2,13}_{6,8}, e^{2,14}_{7,8}, e^{3,12}_{6,9}, e^{4,12}_{7,9}, e^{3,13}_{6,10}, e^{5,13}_{8,10}, e^{3,15}_{9,10}, e^{4,14}_{7,11}, e^{5,14}_{8,11}, e^{4,15}_{9,11}, e^{5,15}_{10,11}\right\} \\ -\mathbf{q}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{6,7}_{12,2}, e^{6,8}_{13,2}, e^{7,8}_{14,2}, e^{6,9}_{12,3}, e^{7,9}_{12,4}, e^{6,10}_{13,3}, e^{8,10}_{13,5}, e^{9,10}_{15,3}, e^{7,11}_{14,4}, e^{8,11}_{14,5}, e^{9,11}_{15,4}, e^{10,11}_{15,5}\right\} \\ +q^{u-\frac{1}{2}}\left\{e^{12,2}_{6,7}, e^{13,2}_{6,8}, e^{14,2}_{7,8}, e^{12,3}_{6,9}, e^{12,4}_{7,9}, e^{13,3}_{6,10}, e^{13,5}_{8,10}, e^{15,3}_{9,10}, e^{14,4}_{7,11}, e^{14,5}_{8,11}, e^{15,4}_{9,11}, e^{15,5}_{10,11}\right\} \\ +\overline{\mathbf{q}}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{7,6}_{2,12}, e^{8,6}_{2,13}, e^{8,7}_{2,14}, e^{9,6}_{3,12}, e^{9,7}_{4,12}, e^{10,6}_{3,13}, e^{10,8}_{5,13}, e^{10,9}_{3,15}, e^{11,7}_{4,14}, e^{11,8}_{5,14}, e^{11,9}_{4,15}, e^{11,10}_{5,15}\right\} \\ -\overline{q}^{u-\frac{1}{2}}\left\{e^{2,12}_{7,6}, e^{2,13}_{8,6}, e^{2,14}_{8,7}, e^{3,12}_{9,6}, e^{4,12}_{9,7}, e^{3,13}_{10,6}, e^{5,13}_{10,8}, e^{3,15}_{10,9}, e^{4,14}_{11,7}, e^{5,14}_{11,8}, e^{4,15}_{11,9}, e^{5,15}_{11,10}\right\}\end{array}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_2^{\frac{1}{2}} U_0 U_1}{S_0^- S_1^- S_2^-} \left\{\begin{matrix} q^{u+1}\left\{\begin{matrix} e^{12,1}_{4,6}, e^{13,1}_{5,6}, e^{14,1}_{5,7}, e^{15,1}_{5,9} \\ e^{4,6}_{12,1}, e^{5,6}_{13,1}, e^{5,7}_{14,1}, e^{5,9}_{15,1} \end{matrix}\right\}, -q^{u}\left\{\begin{matrix} e^{12,1}_{3,7}, e^{13,1}_{3,8}, e^{14,1}_{4,8}, e^{15,1}_{4,10} \\ e^{3,7}_{12,1}, e^{3,8}_{13,1}, e^{4,8}_{14,1}, e^{4,10}_{15,1} \end{matrix}\right\}, q^{u-1}\left\{\begin{matrix} e^{12,1}_{2,9}, e^{13,1}_{2,10}, e^{14,1}_{2,11}, e^{15,1}_{3,11} \\ e^{2,9}_{12,1}, e^{2,10}_{13,1}, e^{2,11}_{14,1}, e^{3,11}_{15,1} \end{matrix}\right\} \\ \overline{q}^{u+1}\left\{\begin{matrix} e^{1,12}_{6,4}, e^{1,13}_{6,5}, e^{1,14}_{7,5}, e^{1,15}_{9,5} \\ e^{6,4}_{1,12}, e^{6,5}_{1,13}, e^{7,5}_{1,14}, e^{9,5}_{1,15} \end{matrix}\right\}, -\overline{q}^{u}\left\{\begin{matrix} e^{1,12}_{7,3}, e^{1,13}_{8,3}, e^{1,14}_{8,4}, e^{1,15}_{10,4} \\ e^{7,3}_{1,12}, e^{8,3}_{1,13}, e^{8,4}_{1,14}, e^{10,4}_{1,15} \end{matrix}\right\}, \overline{q}^{u-1}\left\{\begin{matrix} e^{1,12}_{9,2}, e^{1,13}_{10,2}, e^{1,14}_{11,2}, e^{1,15}_{11,3} \\ e^{9,2}_{1,12}, e^{10,2}_{1,13}, e^{11,2}_{1,14}, e^{11,3}_{1,15} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_1 A_2 A_3^{\frac{1}{2}} U_0}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} q^{3u+\frac{3}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{15,2}} \\ +e^{15,2}_{16,1} \end{matrix}\right\}, q^{3u+\frac{1}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{14,3}} \\ -e^{14,3}_{16,1} \end{matrix}\right\}, q^{3u-\frac{1}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{13,4}} \\ +e^{13,4}_{16,1} \end{matrix}\right\}, q^{3u-\frac{3}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{12,5}} \\ -e^{12,5}_{16,1} \end{matrix}\right\} \\ \overline{q}^{3u+\frac{3}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{1,16}}_{\mathbf{2,15}} \\ -e^{2,15}_{1,16} \end{matrix}\right\}, \overline{q}^{3u+\frac{1}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{1,16}}_{\mathbf{3,14}} \\ +e^{3,14}_{1,16} \end{matrix}\right\}, \overline{q}^{3u-\frac{1}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{1,16}}_{\mathbf{4,13}} \\ -e^{4,13}_{1,16} \end{matrix}\right\}, \overline{q}^{3u-\frac{3}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{1,16}}_{\mathbf{5,12}} \\ e^{5,12}_{1,16} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_3^{\frac{1}{2}} U_0 U_1 U_1}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} q^{u-\frac{3}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{2,15}} \\ -e^{2,15}_{16,1} \end{matrix}\right\}, q^{u-\frac{1}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{3,14}} \\ +e^{3,14}_{16,1} \end{matrix}\right\}, q^{u+\frac{1}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{4,13}} \\ -e^{4,13}_{16,1} \end{matrix}\right\}, q^{3u+\frac{3}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{5,12}} \\ +e^{5,12}_{16,1} \end{matrix}\right\} \\ \overline{q}^{u-\frac{3}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{1,16}}_{\mathbf{15,2}} \\ +e^{15,2}_{1,16} \end{matrix}\right\}, \overline{q}^{u-\frac{1}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{1,16}}_{\mathbf{14,3}} \\ -e^{14,3}_{1,16} \end{matrix}\right\}, \overline{q}^{u+\frac{1}{2}}\left\{\begin{matrix} -\mathbf{e}^{\mathbf{1,16}}_{\mathbf{13,4}} \\ +e^{13,4}_{1,16} \end{matrix}\right\}, \overline{q}^{u+\frac{3}{2}}\left\{\begin{matrix} +\mathbf{e}^{\mathbf{1,16}}_{\mathbf{12,5}} \\ -e^{12,5}_{1,16} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_0^{\frac{1}{2}} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}} U_0 U_1}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} q^{2u-2}\left\{\begin{matrix} e^{16,1}_{11,6}, e^{16,1}_{6,11} \\ e^{11,6}_{16,1}, e^{6,11}_{16,1} \end{matrix}\right\}, -q^{2u+1}\left\{\begin{matrix} e^{16,1}_{10,7}, e^{16,1}_{7,10} \\ e^{10,7}_{16,1}, e^{7,10}_{16,1} \end{matrix}\right\}, q^{2u}\left\{\begin{matrix} e^{16,1}_{9,8}, e^{16,1}_{8,9} \\ e^{8,9}_{16,1}, e^{9,8}_{16,1} \end{matrix}\right\} \\ \overline{q}^{2u-2}\left\{\begin{matrix} e^{11,6}_{1,16}, e^{6,11}_{1,16} \\ e^{1,16}_{11,6}, e^{1,16}_{6,11} \end{matrix}\right\}, -\overline{q}^{2u+1}\left\{\begin{matrix} e^{10,7}_{1,16}, e^{7,10}_{1,16} \\ e^{1,16}_{10,7}, e^{1,16}_{7,10} \end{matrix}\right\}, \overline{q}^{2u}\left\{\begin{matrix} e^{9,8}_{1,16}, e^{8,9}_{1,16} \\ e^{1,16}_{9,8}, e^{1,16}_{8,9} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_1 A_2 {U_0}^2}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} -q^{2u+2}\left\{\begin{matrix} e^{14,3}_{15,2} \\ e^{15,2}_{14,3} \end{matrix}\right\}, q^{2u+1}\left\{\begin{matrix} e^{13,4}_{15,2} \\ e^{15,2}_{13,4} \end{matrix}\right\}, -q^{2u}\left\{\begin{matrix} e^{12,5}_{15,2}, e^{13,4}_{14,3} \\ e^{15,2}_{12,5}, e^{14,3}_{13,4} \end{matrix}\right\}, q^{2u-1}\left\{\begin{matrix} e^{14,3}_{12,5} \\ e^{12,5}_{14,3} \end{matrix}\right\}, -q^{2u-2}\left\{\begin{matrix} e^{13,4}_{12,5} \\ e^{12,5}_{13,4} \end{matrix}\right\} \\ -\overline{q}^{2u+2}\left\{\begin{matrix} e^{3,14}_{2,15} \\ e^{2,15}_{3,14} \end{matrix}\right\}, \overline{q}^{2u+1}\left\{\begin{matrix} e^{4,13}_{2,15} \\ e^{2,15}_{4,13} \end{matrix}\right\}, -\overline{q}^{2u}\left\{\begin{matrix} e^{5,12}_{2,15}, e^{4,13}_{3,14} \\ e^{2,15}_{5,12}, e^{3,14}_{4,13} \end{matrix}\right\}, \overline{q}^{2u-1}\left\{\begin{matrix} e^{5,12}_{3,14} \\ e^{3,14}_{5,12} \end{matrix}\right\}, -\overline{q}^{2u-2}\left\{\begin{matrix} e^{4,13}_{5,12} \\ e^{5,12}_{4,13} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} U_0^2 U_1}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} q^{u}\left\{\begin{matrix} -\mathbf{q}^{\frac{\mathbf{3}}{\mathbf{2}}}\left\{e^{8,9}_{15,2}, e^{10,7}_{14,3}, e^{11,6}_{13,4}\right\}, +\mathbf{q}^{\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{9,8}_{14,3}, e^{7,10}_{15,2}, e^{11,6}_{12,5}\right\}, -\overline{\mathbf{q}}^{\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{9,8}_{13,4}, e^{10,7}_{12,5}, e^{6,11}_{15,2}\right\}, +\overline{\mathbf{q}}^{\frac{\mathbf{3}}{\mathbf{2}}}\left\{e^{8,9}_{12,5}, e^{7,10}_{13,4}, e^{6,11}_{14,3}\right\} \\ +q^{\frac{3}{2}}\left\{e^{15,2}_{8,9}, e^{14,3}_{10,7}, e^{13,4}_{11,6}\right\}, -q^{\frac{1}{2}}\left\{e^{14,3}_{9,8}, e^{15,2}_{7,10}, e^{12,5}_{11,6}\right\}, +\overline{q}^{\frac{1}{2}}\left\{e^{13,4}_{9,8}, e^{12,5}_{10,7}, e^{15,2}_{6,11}\right\}, -\overline{q}^{\frac{3}{2}}\left\{e^{12,5}_{8,9}, e^{13,4}_{7,10}, e^{14,3}_{6,11}\right\} \end{matrix}\right\} \\ \overline{q}^{u}\left\{\begin{matrix} +\overline{\mathbf{q}}^{\frac{\mathbf{3}}{\mathbf{2}}}\left\{e^{9,8}_{2,15}, e^{7,10}_{3,14}, e^{6,11}_{4,13}\right\}, -\overline{\mathbf{q}}^{\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{8,9}_{3,14}, e^{10,7}_{2,15}, e^{6,11}_{5,12}\right\}, +\mathbf{q}^{\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{8,9}_{4,13}, e^{7,10}_{5,12}, e^{11,6}_{2,15}\right\}, -\mathbf{q}^{\frac{\mathbf{3}}{\mathbf{2}}}\left\{e^{9,8}_{5,12}, e^{10,7}_{4,13}, e^{11,6}_{3,14}\right\} \\ -\overline{q}^{\frac{3}{2}}\left\{e^{2,15}_{9,8}, e^{3,14}_{7,10}, e^{4,13}_{6,11}\right\}, +\overline{q}^{\frac{1}{2}}\left\{e^{3,14}_{8,9}, e^{2,15}_{10,7}, e^{5,12}_{6,11}\right\}, -q^{\frac{1}{2}}\left\{e^{4,13}_{8,9}, e^{5,12}_{7,10}, e^{2,15}_{11,6}\right\}, +q^{\frac{3}{2}}\left\{e^{5,12}_{9,8}, e^{4,13}_{10,7}, e^{3,14}_{11,6}\right\} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{U_0}{\Delta^2 S_0^- S_1^- S_2^-} \left\{\begin{matrix} h_1(q)\left\{\begin{matrix} -\mathbf{1}\left\{e^{9,2}_{3,7}, e^{10,2}_{3,8}, e^{7,3}_{4,6}, e^{11,2}_{4,8}, e^{11,3}_{4,10}, e^{8,3}_{5,6}, e^{8,4}_{5,7}, e^{10,4}_{5,9}\right\} \\ +1\left\{e^{3,7}_{9,2}, e^{3,8}_{10,2}, e^{4,6}_{7,3}, e^{4,8}_{11,2}, e^{4,10}_{11,3}, e^{5,6}_{8,3}, e^{5,7}_{8,4}, e^{5,9}_{10,4}\right\} \end{matrix}\right\} \\ h_1(\overline{q})\left\{\begin{matrix} +1\left\{e^{2,9}_{7,3}, e^{2,10}_{8,3}, e^{3,7}_{6,4}, e^{2,11}_{8,4}, e^{3,11}_{10,4}, e^{3,8}_{6,5}, e^{4,8}_{7,5}, e^{4,10}_{9,5}\right\} \\ -\mathbf{1}\left\{e^{7,3}_{2,9}, e^{8,3}_{2,10}, e^{6,4}_{3,7}, e^{8,4}_{2,11}, e^{10,4}_{3,11}, e^{6,5}_{3,8}, e^{7,5}_{4,8}, e^{9,5}_{4,10}\right\} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{U_0}{\Delta^2 S_0^- S_1^- S_2^-} \left\{ qh_1(q)\left\{\begin{matrix} +\mathbf{1}\left\{e^{9,2}_{4,6}, e^{10,2}_{5,6}, e^{11,2}_{5,7}, e^{11,3}_{5,9}\right\} \\ -1\left\{e^{4,6}_{9,2}, e^{5,6}_{10,2}, e^{5,7}_{11,2}, e^{5,9}_{11,3}\right\} \end{matrix}\right\}, \overline{q}h_1(\overline{q})\left\{\begin{matrix} +\mathbf{1}\left\{e^{6,4}_{2,9}, e^{6,5}_{2,10}, e^{7,5}_{2,11}, e^{9,5}_{3,11}\right\} \\ -1\left\{e^{2,9}_{6,4}, e^{2,10}_{6,5}, e^{2,11}_{7,5}, e^{3,11}_{9,5}\right\} \end{matrix}\right\} \right\},$$

$$\frac{A_1^{\frac{1}{2}} A_2 A_3^{\frac{1}{2}} U_0 S_0^+}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} -\mathbf{q}^{\mathbf{2u+1}}\left\{e^{16,2}_{14,6}, e^{16,3}_{15,6}, e^{16,4}_{15,7}, e^{16,5}_{15,8}\right\}, +\mathbf{q}^{\mathbf{2u}}\left\{e^{16,2}_{13,7}, e^{16,3}_{13,9}, e^{16,4}_{14,9}, e^{16,5}_{14,10}\right\}, -\mathbf{q}^{\mathbf{2u-1}}\left\{e^{16,2}_{12,8}, e^{16,3}_{12,10}, e^{16,4}_{12,11}, e^{16,5}_{13,11}\right\} \\ +q^{2u+1}\left\{e^{14,6}_{16,2}, e^{15,6}_{16,3}, e^{15,7}_{16,4}, e^{15,8}_{16,5}\right\}, -q^{2u}\left\{e^{13,7}_{16,2}, e^{13,9}_{16,3}, e^{14,9}_{16,4}, e^{14,10}_{16,5}\right\}, +q^{2u-1}\left\{e^{12,8}_{16,2}, e^{12,10}_{16,3}, e^{12,11}_{16,4}, e^{13,11}_{16,5}\right\} \\ -\overline{\mathbf{q}}^{\mathbf{2u+1}}\left\{e^{6,14}_{2,16}, e^{6,15}_{3,16}, e^{7,15}_{4,16}, e^{8,15}_{5,16}\right\}, +\overline{\mathbf{q}}^{\mathbf{2u}}\left\{e^{7,13}_{2,16}, e^{9,13}_{3,16}, e^{9,14}_{4,16}, e^{10,14}_{5,16}\right\}, -\overline{\mathbf{q}}^{\mathbf{2u-1}}\left\{e^{8,12}_{2,16}, e^{10,12}_{3,16}, e^{11,12}_{4,16}, e^{11,13}_{5,16}\right\} \\ +\overline{q}^{2u+1}\left\{e^{2,16}_{6,14}, e^{3,16}_{6,15}, e^{4,16}_{7,15}, e^{5,16}_{8,15}\right\}, -\overline{q}^{2u}\left\{e^{2,16}_{7,13}, e^{3,16}_{9,13}, e^{4,16}_{9,14}, e^{5,16}_{10,14}\right\}, +\overline{q}^{2u-1}\left\{e^{2,16}_{8,12}, e^{3,16}_{10,12}, e^{4,16}_{11,12}, e^{5,16}_{11,13}\right\} \end{matrix}\right\},$$

$$\frac{A_1^{\frac{1}{2}} A_3^{\frac{1}{2}} U_0 U_1 S_0^+}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} q^{u+1}\left\{\begin{matrix} e^{16,2}_{8,12}, e^{16,3}_{10,12}, e^{16,4}_{11,12}, e^{16,5}_{11,13} \\ e^{8,12}_{16,2}, e^{10,12}_{16,3}, e^{11,12}_{16,4}, e^{11,13}_{16,5} \end{matrix}\right\}, -q^{u}\left\{\begin{matrix} e^{16,2}_{7,13}, e^{16,3}_{9,13}, e^{16,4}_{9,14}, e^{16,5}_{10,14} \\ e^{7,13}_{16,2}, e^{9,13}_{16,3}, e^{9,14}_{16,4}, e^{10,14}_{16,5} \end{matrix}\right\}, q^{u-1}\left\{\begin{matrix} e^{16,2}_{6,14}, e^{16,3}_{6,15}, e^{16,4}_{7,15}, e^{16,5}_{8,15} \\ e^{6,14}_{16,2}, e^{6,15}_{16,3}, e^{7,15}_{16,4}, e^{8,15}_{16,5} \end{matrix}\right\} \\ \overline{q}^{u+1}\left\{\begin{matrix} e^{2,16}_{12,8}, e^{3,16}_{12,10}, e^{4,16}_{12,11}, e^{5,16}_{13,11} \\ e^{12,8}_{2,16}, e^{12,10}_{3,16}, e^{12,11}_{4,16}, e^{13,11}_{5,16} \end{matrix}\right\}, -\overline{q}^{u}\left\{\begin{matrix} e^{2,16}_{13,7}, e^{3,16}_{13,9}, e^{4,16}_{14,9}, e^{5,16}_{14,10} \\ e^{13,7}_{2,16}, e^{13,9}_{3,16}, e^{14,9}_{4,16}, e^{14,10}_{5,16} \end{matrix}\right\}, \overline{q}^{u-1}\left\{\begin{matrix} e^{2,16}_{14,6}, e^{3,16}_{15,6}, e^{4,16}_{15,7}, e^{5,16}_{15,8} \\ e^{14,6}_{2,16}, e^{15,6}_{3,16}, e^{15,7}_{4,16}, e^{15,8}_{5,16} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_2 {U_0}^2 S_0^+}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} -q^{u+1}\left\{\begin{matrix} e^{14,6}_{13,7}, e^{15,6}_{13,9}, e^{15,7}_{14,9}, e^{15,8}_{14,10} \\ e^{13,7}_{14,6}, e^{13,9}_{15,6}, e^{14,9}_{15,7}, e^{14,10}_{15,8} \end{matrix}\right\}, q^{u}\left\{\begin{matrix} e^{14,6}_{12,8}, e^{15,6}_{12,10}, e^{15,7}_{12,11}, e^{15,8}_{13,11} \\ e^{12,8}_{14,6}, e^{12,10}_{15,6}, e^{12,11}_{15,7}, e^{13,11}_{15,8} \end{matrix}\right\}, -q^{u-1}\left\{\begin{matrix} e^{13,7}_{12,8}, e^{13,9}_{12,10}, e^{14,9}_{12,11}, e^{14,10}_{13,11} \\ e^{12,8}_{13,7}, e^{12,10}_{13,9}, e^{12,11}_{14,9}, e^{13,11}_{14,10} \end{matrix}\right\} \\ -\overline{q}^{u+1}\left\{\begin{matrix} e^{6,14}_{7,13}, e^{6,15}_{9,13}, e^{7,15}_{9,14}, e^{8,15}_{10,14} \\ e^{7,13}_{6,14}, e^{9,13}_{6,15}, e^{9,14}_{7,15}, e^{10,14}_{8,15} \end{matrix}\right\}, \overline{q}^{u}\left\{\begin{matrix} e^{6,14}_{8,12}, e^{6,15}_{10,12}, e^{7,15}_{11,12}, e^{8,15}_{11,13} \\ e^{8,12}_{6,14}, e^{10,12}_{6,15}, e^{11,12}_{7,15}, e^{11,13}_{8,15} \end{matrix}\right\}, -\overline{q}^{u-1}\left\{\begin{matrix} e^{7,13}_{8,12}, e^{9,13}_{10,12}, e^{9,14}_{11,12}, e^{10,14}_{11,13} \\ e^{8,12}_{7,13}, e^{10,12}_{9,13}, e^{11,12}_{9,14}, e^{11,13}_{10,14} \end{matrix}\right\} \end{matrix}\right\},$$

$$\frac{A_2^{\frac{1}{2}} A_3^{\frac{1}{2}} U_0 S_0^+ S_1^+}{S_0^- S_1^- S_2^- S_3^-} \left\{\begin{matrix} -\mathbf{q}^{\mathbf{u+\frac{1}{2}}}\left\{e^{16,6}_{13,12}, e^{16,7}_{14,12}, e^{16,8}_{14,13}, e^{16,9}_{15,12}, e^{16,10}_{15,13}, e^{16,11}_{15,14}\right\} \\ +q^{u+\frac{1}{2}}\left\{e^{13,12}_{16,6}, e^{14,12}_{16,7}, e^{14,13}_{16,8}, e^{15,12}_{16,9}, e^{15,13}_{16,10}, e^{15,14}_{16,11}\right\} \\ +\overline{\mathbf{q}}^{\mathbf{u+\frac{1}{2}}}\left\{e^{6,16}_{12,13}, e^{7,16}_{12,14}, e^{8,16}_{13,14}, e^{9,16}_{12,15}, e^{10,16}_{13,15}, e^{11,16}_{14,15}\right\} \\ -\overline{q}^{u+\frac{1}{2}}\left\{e^{12,13}_{6,16}, e^{12,14}_{7,16}, e^{13,14}_{8,16}, e^{12,15}_{9,16}, e^{13,15}_{10,16}, e^{14,15}_{11,16}\right\} \\ +\mathbf{q}^{\mathbf{u-\frac{1}{2}}}\left\{e^{16,6}_{12,13}, e^{16,7}_{12,14}, e^{16,8}_{13,14}, e^{16,9}_{12,15}, e^{16,10}_{13,15}, e^{16,11}_{14,15}\right\} \\ -q^{u-\frac{1}{2}}\left\{e^{12,13}_{16,6}, e^{12,14}_{16,7}, e^{13,14}_{16,8}, e^{12,15}_{16,9}, e^{13,15}_{16,10}, e^{14,15}_{16,11}\right\} \\ -\overline{\mathbf{q}}^{\mathbf{u-\frac{1}{2}}}\left\{e^{6,16}_{13,12}, e^{7,16}_{14,12}, e^{8,16}_{14,13}, e^{9,16}_{15,12}, e^{10,16}_{15,13}, e^{11,16}_{15,14}\right\} \\ +\overline{q}^{u-\frac{1}{2}}\left\{e^{13,12}_{6,16}, e^{14,12}_{7,16}, e^{14,13}_{8,16}, e^{15,12}_{9,16}, e^{15,13}_{10,16}, e^{15,14}_{11,16}\right\} \end{matrix}\right\},$$

$$\frac{A_1^{\frac{1}{2}}\,A_2^{\frac{1}{2}}\,U_0}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{+\mathbf{q}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\,\mathbf{h_2(q)}\left\{q e^{11,6}_{15,2}, \overline{q} e^{9,8}_{15,2}\right\},\ -\overline{\mathbf{q}}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\,\mathbf{h_2}(\overline{\mathbf{q}})\left\{\overline{q} e^{6,11}_{2,15}, q e^{8,9}_{2,15}\right\}\right\},$$

$$\frac{A_1^{\frac{1}{2}}\,A_2^{\frac{1}{2}}\,U_0}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l} -\overline{q}^{\,\frac{1}{2}}\,h_2(\overline{q})\left\{\overline{q}^u\left\{\begin{array}{c}-\mathbf{e}^{\mathbf{7,10}}_{\mathbf{2,15}}, -\mathbf{e}^{\mathbf{6,11}}_{\mathbf{3,14}}\\ +e^{2,15}_{7,10}, +e^{3,14}_{6,11}\end{array}\right\},\ q^u\left\{\begin{array}{c}-\mathbf{e}^{\mathbf{7,10}}_{\mathbf{12,5}}, -\mathbf{e}^{\mathbf{6,11}}_{\mathbf{13,4}}\\ +e^{12,5}_{7,10}, +e^{13,4}_{6,11}\end{array}\right\}\right\}\\ +q^{\frac{1}{2}}\,h_2(q)\left\{q^u\left\{\begin{array}{c}-\mathbf{e}^{\mathbf{10,7}}_{\mathbf{15,2}}, -\mathbf{e}^{\mathbf{11,6}}_{\mathbf{14,3}}\\ +e^{15,2}_{10,7}, +e^{14,3}_{11,6}\end{array}\right\},\ \overline{q}^u\left\{\begin{array}{c}-\mathbf{e}^{\mathbf{10,7}}_{\mathbf{5,12}}, -\mathbf{e}^{\mathbf{11,6}}_{\mathbf{4,13}}\\ +e^{5,12}_{10,7}, +e^{4,13}_{11,6}\end{array}\right\}\right\}\end{array}\right\},$$

$$\frac{A_1^{\frac{1}{2}}\,A_2^{\frac{1}{2}}\,U_0}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{+\overline{q}^{\,u-\frac{1}{2}}\,h_2(q)\left\{q\left\{\begin{array}{c}+\mathbf{e}^{\mathbf{11,6}}_{\mathbf{5,12}}\\ -e^{5,12}_{11,6}\end{array}\right\},\ \overline{q}\left\{\begin{array}{c}+\mathbf{e}^{\mathbf{8,9}}_{\mathbf{5,12}}\\ -e^{5,12}_{8,9}\end{array}\right\}\right\},\ -q^{\,u-\frac{1}{2}}\,h_2(\overline{q})\left\{\overline{q}\left\{\begin{array}{c}+\mathbf{e}^{\mathbf{6,11}}_{\mathbf{12,5}}\\ -e^{12,5}_{6,11}\end{array}\right\},\ q\left\{\begin{array}{c}+\mathbf{e}^{\mathbf{9,8}}_{\mathbf{12,5}}\\ -e^{12,5}_{9,8}\end{array}\right\}\right\}\right\},$$

$$\frac{h_3(q)A_1^{\frac{1}{2}}\,A_2^{\frac{1}{2}}\,U_0}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l}-\mathbf{q}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{8,9}_{14,3}, e^{10,7}_{13,4}\right\},\ +\mathbf{q}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{7,10}_{14,3}, e^{8,9}_{13,4}\right\},\ +\overline{\mathbf{q}}^{\mathbf{u}+\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{9,8}_{3,14}, e^{7,10}_{4,13}\right\},\ -\overline{\mathbf{q}}^{\mathbf{u}-\frac{\mathbf{1}}{\mathbf{2}}}\left\{e^{10,7}_{3,14}, e^{9,8}_{4,13}\right\}\\ +q^{u+\frac{1}{2}}\left\{e^{14,3}_{8,9}, e^{13,4}_{10,7}\right\},\ -q^{u-\frac{1}{2}}\left\{e^{14,3}_{7,10}, e^{13,4}_{8,9}\right\},\ -\overline{q}^{u+\frac{1}{2}}\left\{e^{3,14}_{9,8}, e^{4,13}_{7,10}\right\},\ +\overline{q}^{u-\frac{1}{2}}\left\{e^{3,14}_{10,7}, e^{4,13}_{9,8}\right\}\end{array}\right\},$$

$$\frac{{U_0}^2}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l}-\overline{q}h_4(q)\left\{\begin{array}{c}e^{7,10}_{11,6}, e^{8,9}_{10,7}, e^{9,8}_{10,7}\\ e^{11,6}_{7,10}, e^{10,7}_{8,9}, e^{10,7}_{9,8}\end{array}\right\},\ h_4(q)\left\{\begin{array}{c}e^{9,8}_{11,6}, e^{11,6}_{8,9}\\ e^{11,6}_{9,8}, e^{8,9}_{11,6}\end{array}\right\},\ -qh_4(q)\left\{\begin{array}{c}e^{11,6}_{10,7}\\ e^{10,7}_{11,6}\end{array}\right\}\\ -qh_4(\overline{q})\left\{\begin{array}{c}e^{10,7}_{6,11}, e^{9,8}_{7,10}, e^{8,9}_{7,10}\\ e^{6,11}_{10,7}, e^{7,10}_{8,9}, e^{7,10}_{9,8}\end{array}\right\},\ h_4(\overline{q})\left\{\begin{array}{c}e^{8,9}_{6,11}, e^{6,11}_{9,8}\\ e^{6,11}_{8,9}, e^{9,8}_{6,11}\end{array}\right\},\ -\overline{q}h_4(\overline{q})\left\{\begin{array}{c}e^{6,11}_{7,10}\\ e^{7,10}_{6,11}\end{array}\right\}\end{array}\right\},$$

$$\frac{U_0U_1}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l}-\overline{q}h_5(q)\left\{\begin{array}{c}e^{3,14}_{15,2}, e^{5,12}_{13,4}, e^{4,13}_{14,3}\\ e^{15,2}_{3,14}, e^{13,4}_{5,12}, e^{14,3}_{4,13}\end{array}\right\},\ h_5(q)\left\{\begin{array}{c}e^{5,12}_{14,3}, e^{4,13}_{15,2}\\ e^{14,3}_{5,12}, e^{15,2}_{4,13}\end{array}\right\},\ -qh_5(q)\left\{\begin{array}{c}e^{5,12}_{15,2}\\ e^{15,2}_{5,12}\end{array}\right\}\\ -qh_5(\overline{q})\left\{\begin{array}{c}e^{2,15}_{14,3}, e^{4,13}_{12,5}, e^{3,14}_{13,4}\\ e^{14,3}_{2,15}, e^{12,5}_{4,13}, e^{13,4}_{3,14}\end{array}\right\},\ h_5(\overline{q})\left\{\begin{array}{c}e^{2,15}_{13,4}, e^{3,14}_{12,5}\\ e^{13,4}_{2,15}, e^{12,5}_{3,14}\end{array}\right\},\ -\overline{q}h_5(\overline{q})\left\{\begin{array}{c}e^{2,15}_{12,5}\\ e^{12,5}_{2,15}\end{array}\right\}\end{array}\right\},$$

$$\frac{U_0S_0^+}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l}h_6(q)\left\{\begin{array}{l}\mathbf{-1}\left\{e^{7,13}_{14,6}, e^{8,12}_{13,7}, e^{9,13}_{15,6}, e^{9,14}_{15,7}, e^{10,12}_{13,9}, e^{10,14}_{15,8}, e^{11,12}_{14,9}, e^{11,13}_{14,10}\right\}\\ +1\left\{e^{14,6}_{7,13}, e^{13,7}_{8,12}, e^{15,6}_{9,13}, e^{15,7}_{9,14}, e^{13,9}_{10,12}, e^{15,8}_{10,14}, e^{14,9}_{11,12}, e^{14,10}_{11,13}\right\}\end{array}\right\}\\ h_6(\overline{q})\left\{\begin{array}{l}+1\left\{e^{13,7}_{6,14}, e^{12,8}_{7,13}, e^{13,9}_{6,15}, e^{14,9}_{7,15}, e^{12,10}_{9,13}, e^{14,10}_{8,15}, e^{12,11}_{9,14}, e^{13,11}_{10,14}\right\}\\ \mathbf{-1}\left\{e^{6,14}_{13,7}, e^{7,13}_{12,8}, e^{6,15}_{13,9}, e^{7,15}_{14,9}, e^{9,13}_{12,10}, e^{8,15}_{14,10}, e^{9,14}_{12,11}, e^{10,14}_{13,11}\right\}\end{array}\right\}\end{array}\right\},$$

$$\frac{U_0S_0^+}{\Delta^2 S_0^- \,S_1^- \,S_2^- \,S_3^-}\left\{\begin{array}{l}qh_6(q)\left\{\begin{array}{l}\mathbf{+1}\left\{e^{8,12}_{14,6}, e^{10,12}_{15,6}, e^{11,12}_{15,7}, e^{11,13}_{15,8}\right\}\\ -1\left\{e^{14,6}_{8,12}, e^{15,6}_{10,12}, e^{15,7}_{11,12}, e^{15,8}_{11,13}\right\}\end{array}\right\}\\ \overline{q}h_6(\overline{q})\left\{\begin{array}{l}-1\left\{e^{12,8}_{6,14}, e^{12,10}_{6,15}, e^{12,11}_{7,15}, e^{13,11}_{8,15}\right\}\\ \mathbf{+1}\left\{e^{6,14}_{12,8}, e^{6,15}_{12,10}, e^{7,15}_{12,11}, e^{8,15}_{13,11}\right\}\end{array}\right\}\end{array}\right\},$$

where:

$$\begin{aligned}
g_1(q) &= -2q + (q^{1+2\alpha} + \overline{q}^{1+2\alpha}) + q^{2u}(q-\overline{q})\\
g_2(q) &= \overline{q}^u(-2 + (q^{2+2\alpha} + \overline{q}^{2+2\alpha}) + (q^{2u} + \overline{q}^{2u}) - (q^{2u-2} + \overline{q}^{2u-2}))\\
g_3(q) &= -2q^2 + (q^{2\alpha+2} + \overline{q}^{2\alpha+2}) + q^{2u}(q^2-\overline{q}^2)\\
g_4(q) &= -2q + (q^{3+2\alpha} + \overline{q}^{3+2\alpha}) + q^{2u}(q-\overline{q})\\
g_5(q) &= -2q^2 + (q^{4+2\alpha} + \overline{q}^{4+2\alpha}) + q^{2u}(q^2-\overline{q}^2)\\
g_6(q) &= -2q^3 + (q^{3+2\alpha} + \overline{q}^{3+2\alpha}) + q^{2u}(q^3-\overline{q}^3)\\
g_7(q) &= -2q + (q^{3+2\alpha} + \overline{q}^{3+2\alpha}) + q(q^{2u} + \overline{q}^{2u}) - (q^{2u-3} + \overline{q}^{2u-3})\\
g_8(q) &= q^u(-2 + (q^{4+2\alpha} + \overline{q}^{4+2\alpha}) + (q^{2u} + \overline{q}^{2u}) - (q^{2u-2} + \overline{q}^{2u-2}))\\
g_9(q) &= -2q + (q^{5+2\alpha} - \overline{q}^{5+2\alpha}) + q^{2u}(q-\overline{q})\\
g_{10}(q) &= 1 + 2q^2 + 3q^4 - 2\overline{q}^{2+2\alpha} - 2\overline{q}^{2\alpha} - 2q^{4+2\alpha} - 2q^{6+2\alpha} + (q^{6+4\alpha} + \overline{q}^{6+4\alpha}) + 2q^{2u} - q^{4u} - q^{2u}(q^2-\overline{q}^2)\\
&\quad -2q^{4+2u} + q^{4u-4} + q^{2u}(q^{4+2\alpha} - \overline{q}^{4+2\alpha}) + q^{4u}(q^2-\overline{q}^2) + q^{2u}(q^{6+2\alpha} - \overline{q}^{6+2\alpha}) - q^{2u}(q^{2\alpha} - \overline{q}^{2\alpha})\\
&\quad -q^{2u}(q^{2+2\alpha} - \overline{q}^{2+2\alpha})\\
g_{11}(q) &= 3 - \overline{q}^2 + 5q^2 - q^4 + (q^{6+4\alpha} + \overline{q}^{6+4\alpha}) - 4\overline{q}^{2+2\alpha} - 4q^{4+2\alpha} + 2q^{-2-2\alpha+2u} + 2q^{4+2\alpha+2u}\\
&\quad -(q^{2\alpha+2u} + \overline{q}^{2\alpha+2u}) - (q^{2+2\alpha+2u} - \overline{q}^{2+2\alpha+2u}) + (q^{4+2\alpha-2u} - \overline{q}^{4+2\alpha-2u}) - (q^{6+2\alpha-2u} + \overline{q}^{6+2\alpha-2u})\\
&\quad -2q^2(q^{2u} + \overline{q}^{2u}) + 2q^{4-2u} - 2\overline{q}^{2-4u} - 2q^{2u} + q^{4u} + \overline{q}^{4-4u} + 4\overline{q}^{2-2u}\\
g_{12}(q) &= 6 + (q^2 + \overline{q}^2) - (q^4 + \overline{q}^4) - 2(q^{2+2\alpha} + \overline{q}^{2+2\alpha}) - 2(q^{4+2\alpha} + \overline{q}^{4+2\alpha}) + (q^{6+4\alpha} + \overline{q}^{6+4\alpha}) + 2q^{4-2u},
\end{aligned}$$

and:

$$\begin{aligned}
h_1(q) &= -q(q+\overline{q}) + (q^{2+2\alpha} + \overline{q}^{2+2\alpha}) + q^{2u-1}(q-\overline{q})\\
h_2(q) &= -(q^{3+2\alpha} + \overline{q}^{3+2\alpha}) - q^{u+1}(q^u - \overline{q}^u) + q^u(q^{3+u} + \overline{q}^{3+u})\\
h_3(q) &= (q+\overline{q}) - (q^{3+2\alpha} + \overline{q}^{3+2\alpha}) - (q^{1-2u} + \overline{q}^{1-2u}) + (q^{3-2u} + \overline{q}^{3-2u})\\
h_4(q) &= -2q^2 + q(\overline{q}^{3+2\alpha} + q^{3+2\alpha}) + q^{2u-1}(q-\overline{q})
\end{aligned}$$

$$
\begin{aligned}
h_5(q) &= q^{2u-1}(q-\overline{q}) - q^2(q^2-\overline{q}^2) + q(q^{3+2\alpha}+\overline{q}^{3+2\alpha}) \\
h_6(q) &= -(q^{4+2\alpha}+\overline{q}^{4+2\alpha}) - q^u(q^u-\overline{q}^u) + q^u(q^{2-u}+\overline{q}^{2-u}).
\end{aligned}
$$

$\check{R}^4$ has 758 nonzero components:

$$1\left\{e^{1,1}_{1,1}\right\}, \qquad q^{4\alpha+2}\left\{e^{6,6}_{6,6}, e^{7,7}_{7,7}, e^{8,8}_{8,8}, e^{9,9}_{9,9}, e^{10,10}_{10,10}, e^{11,11}_{11,11}\right\}, \qquad q^{8\alpha+12}\left\{e^{16,16}_{16,16}\right\},$$

$$-q^{2\alpha}\left\{e^{2,2}_{2,2}, e^{3,3}_{3,3}, e^{4,4}_{4,4}, e^{5,5}_{5,5},\right\}, \qquad -q^{6\alpha+6}\left\{e^{12,12}_{12,12}, e^{13,13}_{13,13}, e^{14,14}_{14,14}, e^{15,15}_{15,15}\right\},$$

$$-\Delta q^{\alpha}A_0\left\{e^{2,1}_{2,1}, e^{3,1}_{3,1}, e^{4,1}_{4,1}, e^{5,1}_{5,1}\right\}, \qquad -\Delta^3 q^{3\alpha+3}A_0A_1A_2\left\{e^{12,1}_{12,1}, e^{13,1}_{13,1}, e^{14,1}_{14,1}, e^{15,1}_{15,1}\right\},$$

$$\Delta q^{7\alpha+9}A_3\left\{e^{16,15}_{16,15}, e^{16,14}_{16,14}, e^{16,13}_{16,13}, e^{16,12}_{16,12}\right\}, \qquad \Delta^3 q^{5\alpha+6}A_1A_2A_3\left\{e^{16,5}_{16,5}, e^{16,4}_{16,4}, e^{16,3}_{16,3}, e^{16,2}_{16,2}\right\},$$

$$\Delta^2 q^{2\alpha+1}A_0A_1\left\{e^{6,1}_{6,1}, e^{7,1}_{7,1}, e^{8,1}_{8,1}, e^{9,1}_{9,1}, e^{10,1}_{10,1}, e^{11,1}_{11,1}\right\},$$

$$\Delta^2 q^{6\alpha+7}A_2A_3\left\{e^{16,11}_{16,11}, e^{16,10}_{16,10}, e^{16,9}_{16,9}, e^{16,8}_{16,8}, e^{16,6}_{16,6}, e^{16,7}_{16,7}\right\},$$

$$\Delta q^{2\alpha+1}\left\{e^{3,2}_{3,2}, e^{4,2}_{4,2}, e^{5,2}_{5,2}, e^{4,3}_{4,3}, e^{5,3}_{5,3}, e^{5,4}_{5,4}\right\}, \qquad \Delta q^{6\alpha+1}\left\{e^{15,14}_{15,14}, e^{15,13}_{15,13}, e^{15,12}_{15,12}, e^{14,13}_{14,13}, e^{14,12}_{14,12}, e^{13,12}_{13,12}\right\},$$

$$\Delta q^{6\alpha+7}A_0\left\{e^{6,2}_{6,2}, e^{6,3}_{6,3}, e^{7,2}_{7,2}, e^{7,4}_{7,4}, e^{8,2}_{8,2}, e^{8,5}_{8,5}, e^{9,3}_{9,3}, e^{9,4}_{9,4}, e^{10,3}_{10,3}, e^{10,5}_{10,5}, e^{11,4}_{11,4}, e^{11,5}_{11,5}\right\},$$

$$-\Delta q^{5\alpha+4}A_2\left\{e^{12,6}_{12,6}, e^{12,7}_{12,7}, e^{12,9}_{12,9}, e^{13,6}_{13,6}, e^{13,8}_{13,8}, e^{13,10}_{13,10}, e^{14,7}_{14,7}, e^{14,8}_{14,8}, e^{14,11}_{14,11}, e^{15,9}_{15,9}, e^{15,10}_{15,10}, e^{15,11}_{15,11}\right\},$$

$$-\Delta^2 q^{4\alpha+3}A_1A_2\left\{e^{12,2}_{12,2}, e^{12,3}_{12,3}, e^{12,4}_{12,4}, e^{13,2}_{13,2}, e^{13,3}_{13,3}, e^{13,5}_{13,5}, e^{14,2}_{14,2}, e^{14,4}_{14,4}, e^{14,5}_{14,5}, e^{15,3}_{15,3}, e^{15,4}_{15,4}, e^{15,5}_{15,5}\right\},$$

$$-\Delta q^{4\alpha+3}\left\{e^{7,6}_{7,6}, e^{8,6}_{8,6}, e^{8,7}_{8,7}, e^{9,6}_{9,6}, e^{9,7}_{9,7}, e^{10,6}_{10,6}, e^{10,8}_{10,8}, e^{10,9}_{10,9}, e^{11,7}_{11,7}, e^{11,8}_{11,8}, e^{11,9}_{11,9}, e^{11,10}_{11,10}\right\},$$

$$-\Delta^2 q^{3\alpha+2}A_1\left\{e^{7,3}_{7,3}, e^{8,3}_{8,3}, e^{8,4}_{8,4}, e^{10,4}_{10,4}\right\}, \qquad \Delta^2 q^{5\alpha+5}A_2\left\{e^{14,10}_{14,10}, e^{14,9}_{14,9}, e^{13,9}_{13,9}, e^{13,7}_{13,7}\right\},$$

$$-\Delta q^{3\alpha+3}(q^2-\overline{q}^2)A_0\left\{e^{9,2}_{9,2}, e^{10,2}_{10,2}, e^{11,2}_{11,2}, e^{11,3}_{11,3}\right\}, \qquad \Delta q^{5\alpha+6}(q^2-\overline{q}^2)A_2\left\{e^{15,8}_{15,8}, e^{15,7}_{15,7}, e^{15,6}_{15,6}, e^{14,6}_{14,6}\right\},$$

$$\Delta^2 q^{4\alpha+4}\left\{e^{10,7}_{10,7}\right\}, \qquad \Delta^2 q^{4\alpha+5}(q+\overline{q})\left\{e^{11,6}_{11,6}\right\}, \qquad \Delta^3 q^{4\alpha+6}(q^2+1+\overline{q}^2)A_1A_2\left\{e^{15,2}_{15,2}\right\},$$

$$\Delta^2 q^{4\alpha+5}(q^2-\overline{q}^2)A_1A_2\left\{e^{14,3}_{14,3}\right\}, \qquad \Delta^3 q^{4\alpha+4}A_1A_2\left\{e^{13,4}_{13,4}\right\}, \qquad \Delta^4 q^{4\alpha+6}A_0A_1A_2A_3\left\{e^{16,1}_{16,1}\right\},$$

$$q^{\alpha}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{1,2}_{2,1}, e^{1,3}_{3,1}, e^{1,4}_{4,1}, e^{1,5}_{5,1}\right\} \\ +1\left\{e^{2,1}_{1,2}, e^{3,1}_{1,3}, e^{4,1}_{1,4}, e^{5,1}_{1,5}\right\} \end{array}\right\}, \qquad q^{3\alpha}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{1,12}_{12,1}, e^{1,13}_{13,1}, e^{1,14}_{14,1}, e^{1,15}_{15,1}\right\} \\ +1\left\{e^{12,1}_{1,12}, e^{13,1}_{1,13}, e^{14,1}_{1,14}, e^{15,1}_{1,15}\right\} \end{array}\right\}, \qquad q^{4\alpha}\left\{\begin{array}{l} e^{1,16}_{16,1} \\ e^{16,1}_{1,16} \end{array}\right\},$$

$$q^{2\alpha}\left\{\begin{array}{l} e^{1,6}_{6,1}, e^{1,7}_{7,1}, e^{1,8}_{8,1}, e^{1,9}_{9,1}, e^{1,10}_{10,1}, e^{1,11}_{11,1} \\ e^{6,1}_{1,6}, e^{7,1}_{1,7}, e^{8,1}_{1,8}, e^{9,1}_{1,9}, e^{10,1}_{1,10}, e^{11,1}_{1,11} \end{array}\right\}, \qquad -q^{2\alpha+1}\left\{\begin{array}{l} e^{2,3}_{3,2}, e^{2,4}_{4,2}, e^{2,5}_{5,2}, e^{3,4}_{4,3}, e^{3,5}_{5,3}, e^{4,5}_{5,4} \\ e^{3,2}_{2,3}, e^{4,2}_{2,4}, e^{5,2}_{2,5}, e^{4,3}_{3,4}, e^{5,3}_{3,5}, e^{5,4}_{4,5} \end{array}\right\},$$

$$q^{3\alpha+1}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,2}_{2,6}, e^{7,2}_{2,7}, e^{8,2}_{2,8}, e^{6,3}_{3,6}, e^{9,3}_{3,9}, e^{10,3}_{3,10}, e^{7,4}_{4,7}, e^{9,4}_{4,9}, e^{11,4}_{4,11}, e^{8,5}_{5,8}, e^{10,5}_{5,10}, e^{11,5}_{5,11}\right\} \\ +1\left\{e^{2,6}_{6,2}, e^{2,7}_{7,2}, e^{2,8}_{8,2}, e^{3,6}_{6,3}, e^{3,9}_{9,3}, e^{3,10}_{10,3}, e^{4,7}_{7,4}, e^{4,9}_{9,4}, e^{4,11}_{11,4}, e^{5,8}_{8,5}, e^{5,10}_{10,5}, e^{5,11}_{11,5}\right\} \end{array}\right\}$$

$$q^{3\alpha+2}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,4}_{4,6}, e^{6,5}_{5,6}, e^{7,3}_{3,7}, e^{7,5}_{5,7}, e^{8,3}_{3,8}, e^{8,4}_{4,8}, e^{9,2}_{2,9}, e^{9,5}_{5,9}, e^{10,2}_{2,10}, e^{10,4}_{4,10}, e^{11,2}_{2,11}, e^{11,3}_{3,11}\right\} \\ +1\left\{e^{4,6}_{6,4}, e^{5,6}_{6,5}, e^{3,7}_{7,3}, e^{5,7}_{7,5}, e^{3,8}_{8,3}, e^{4,8}_{8,4}, e^{2,9}_{9,2}, e^{5,9}_{9,5}, e^{2,10}_{10,2}, e^{4,10}_{10,4}, e^{2,11}_{11,2}, e^{3,11}_{11,3}\right\} \end{array}\right\},$$

$$-q^{4\alpha+2}\left\{\begin{array}{l} e^{2,12}_{12,2}, e^{2,13}_{13,2}, e^{2,14}_{14,2}, e^{3,12}_{12,3}, e^{3,13}_{13,3}, e^{3,15}_{15,3}, e^{4,12}_{12,4}, e^{4,14}_{14,4}, e^{4,15}_{15,4}, e^{5,13}_{13,5}, e^{5,14}_{14,5}, e^{5,15}_{15,5} \\ e^{12,2}_{2,12}, e^{13,2}_{2,13}, e^{14,2}_{2,14}, e^{12,3}_{3,12}, e^{13,3}_{3,13}, e^{15,3}_{3,15}, e^{12,4}_{4,12}, e^{14,4}_{4,14}, e^{13,5}_{5,13}, e^{14,5}_{5,14}, e^{15,4}_{4,15}, e^{15,5}_{5,15} \end{array}\right\},$$

$$q^{4\alpha+3}\left\{\begin{array}{l} e^{7,11}_{11,7}, e^{8,11}_{11,8}, e^{9,11}_{11,9}, e^{10,11}_{11,10}, e^{8,10}_{10,8}, e^{9,10}_{10,9}, e^{7,8}_{8,7}, e^{7,9}_{9,7}, e^{6,7}_{7,6}, e^{6,8}_{8,6}, e^{6,9}_{9,6}, e^{6,10}_{10,6} \\ e^{11,7}_{7,11}, e^{11,8}_{8,11}, e^{11,9}_{9,11}, e^{11,10}_{10,11}, e^{10,8}_{8,10}, e^{10,9}_{9,10}, e^{8,7}_{7,8}, e^{9,7}_{7,9}, e^{7,6}_{6,7}, e^{8,6}_{6,8}, e^{9,6}_{6,9}, e^{10,6}_{6,10} \end{array}\right\},$$

$$q^{5\alpha+3}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,5}_{5,16}, e^{16,4}_{4,16}, e^{16,3}_{3,16}, e^{16,2}_{2,16}\right\} \\ +1\left\{e^{5,16}_{16,5}, e^{4,16}_{16,4}, e^{3,16}_{16,3}, e^{2,16}_{16,2}\right\} \end{array}\right\}, \qquad -q^{4\alpha+3}\left\{\begin{array}{l} e^{2,15}_{15,2}, e^{3,14}_{14,3}, e^{4,13}_{13,4}, e^{5,12}_{12,5} \\ e^{15,2}_{2,15}, e^{14,3}_{3,14}, e^{13,4}_{4,13}, e^{12,5}_{5,12} \end{array}\right\},$$

$$q^{5\alpha+4}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,12}_{12,6}, e^{6,13}_{13,6}, e^{7,12}_{12,7}, e^{7,14}_{14,7}, e^{8,13}_{13,8}, e^{8,14}_{14,8}, e^{9,12}_{12,9}, e^{9,15}_{15,9}, e^{10,13}_{13,10}, e^{10,15}_{15,10}, e^{11,14}_{14,11}, e^{11,15}_{15,11}\right\} \\ +1\left\{e^{12,6}_{6,12}, e^{13,6}_{6,13}, e^{12,7}_{7,12}, e^{14,7}_{7,14}, e^{13,8}_{8,13}, e^{14,8}_{8,14}, e^{12,9}_{9,12}, e^{15,9}_{9,15}, e^{13,10}_{10,13}, e^{15,10}_{10,15}, e^{14,11}_{11,14}, e^{15,11}_{11,15}\right\} \end{array}\right\},$$

$$q^{5\alpha+5}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,14}_{14,6}, e^{6,15}_{15,6}, e^{7,13}_{13,7}, e^{7,15}_{15,7}, e^{8,12}_{12,8}, e^{8,15}_{15,8}, e^{9,13}_{13,9}, e^{9,14}_{14,9}, e^{10,12}_{12,10}, e^{10,14}_{14,10}, e^{11,12}_{12,11}, e^{11,13}_{13,11}\right\} \\ +1\left\{e^{14,6}_{6,14}, e^{15,6}_{6,15}, e^{13,7}_{7,13}, e^{15,7}_{7,15}, e^{12,8}_{8,12}, e^{15,8}_{8,15}, e^{13,9}_{9,13}, e^{14,9}_{9,14}, e^{12,10}_{10,12}, e^{14,10}_{10,14}, e^{12,11}_{11,12}, e^{13,11}_{11,13}\right\} \end{array}\right\},$$

$$q^{6\alpha+6}\left\{\begin{array}{l} e^{6,16}_{16,6}, e^{7,16}_{16,7}, e^{8,16}_{16,8}, e^{9,16}_{16,9}, e^{10,16}_{16,10}, e^{11,16}_{16,11} \\ e^{16,6}_{6,16}, e^{16,7}_{7,16}, e^{16,8}_{8,16}, e^{16,9}_{9,16}, e^{16,10}_{10,16}, e^{16,11}_{11,16} \end{array}\right\}, \qquad -q^{6\alpha+7}\left\{\begin{array}{l} e^{12,13}_{13,12}, e^{12,14}_{14,12}, e^{13,14}_{14,13}, e^{12,15}_{15,12}, e^{13,15}_{15,13}, e^{14,15}_{15,14} \\ e^{13,12}_{12,13}, e^{14,12}_{12,14}, e^{14,13}_{13,14}, e^{15,12}_{12,15}, e^{15,13}_{13,15}, e^{15,14}_{14,15} \end{array}\right\},$$

$$q^{7\alpha+9}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,15}_{15,16}, e^{16,14}_{14,16}, e^{16,13}_{13,16}, e^{16,12}_{12,16}\right\} \\ +1\left\{e^{15,16}_{16,15}, e^{14,16}_{16,14}, e^{13,16}_{16,13}, e^{12,16}_{16,12}\right\} \end{array}\right\}, \qquad q^{4\alpha+4}\left\{\begin{array}{l} e^{6,11}_{11,6}, e^{7,10}_{10,7}, e^{8,9}_{9,8} \\ e^{11,6}_{6,11}, e^{10,7}_{7,10}, e^{9,8}_{8,9} \end{array}\right\},$$

$$\Delta q^{3\alpha+3}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{9,2}_{4,6}, e^{10,2}_{5,6}, e^{11,2}_{5,7}, e^{11,3}_{5,9}\right\} \\ +\left\{e^{4,6}_{9,2}, e^{5,6}_{10,2}, e^{5,7}_{11,2}, e^{5,9}_{11,3}\right\} \end{array}\right\}, \qquad \Delta q^{3\alpha+2}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{7,3}_{4,6}, e^{8,3}_{5,6}, e^{8,4}_{5,7}, e^{9,2}_{3,7}, e^{10,2}_{3,8}, e^{10,4}_{5,9}, e^{11,2}_{4,8}, e^{11,3}_{4,10}\right\} \\ -1\left\{e^{4,6}_{7,3}, e^{5,6}_{8,3}, e^{5,7}_{8,4}, e^{3,7}_{9,2}, e^{3,8}_{10,2}, e^{5,9}_{10,4}, e^{4,8}_{11,2}, e^{4,10}_{11,3}\right\} \end{array}\right\},$$

$$\Delta q^{4\alpha+5}\left\{-\overline{q}\left\{\begin{array}{l} e^{4,13}_{15,2}, e^{5,12}_{14,3}, e^{7,10}_{11,6}, e^{8,9}_{10,7}, e^{9,8}_{10,7} \\ e^{15,2}_{4,13}, e^{14,3}_{5,12}, e^{11,6}_{7,10}, e^{10,7}_{8,9}, e^{10,7}_{9,8}\end{array}\right\}, \left\{\begin{array}{l} e^{5,12}_{15,2}, e^{8,9}_{11,6}, e^{9,8}_{11,6} \\ e^{15,2}_{5,12}, e^{11,6}_{8,9}, e^{11,6}_{9,8}\end{array}\right\}, -q\left\{\begin{array}{l} e^{10,7}_{11,6}\\ e^{11,6}_{10,7}\end{array}\right\}\right\},$$

$$\Delta q^{5\alpha+5}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{7,13}_{14,6}, e^{8,12}_{13,7}, e^{9,13}_{15,6}, e^{9,14}_{15,7}, e^{10,12}_{13,9}, e^{10,14}_{15,8}, e^{11,12}_{14,9}, e^{11,13}_{14,10}\right\} \\ -1\left\{e^{14,6}_{7,13}, e^{13,7}_{8,12}, e^{15,6}_{9,13}, e^{15,7}_{9,14}, e^{13,9}_{10,12}, e^{15,8}_{10,14}, e^{14,9}_{11,12}, e^{14,10}_{11,13}\right\}\end{array}\right\}, \qquad \Delta q^{4\alpha+3}\left\{\begin{array}{l} e^{3,14}_{15,2}, e^{4,13}_{14,3}, e^{5,12}_{13,4} \\ e^{15,2}_{3,14}, e^{14,3}_{4,13}, e^{13,4}_{5,12}\end{array}\right\},$$

$$\Delta q^{5\alpha+6}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{8,12}_{14,6}, e^{10,12}_{15,6}, e^{11,12}_{15,7}, e^{11,13}_{15,8}\right\} \\ +1\left\{e^{14,6}_{8,12}, e^{15,6}_{10,12}, e^{15,7}_{11,12}, e^{15,8}_{11,13}\right\}\end{array}\right\},$$

$$\Delta q^{2\alpha+1} A_0^{\frac{1}{2}} A_1^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{q}^{\boldsymbol{\frac{1}{2}}}\left\{e^{6,1}_{3,2}, e^{7,1}_{4,2}, e^{8,1}_{5,2}, e^{9,1}_{4,3}, e^{10,1}_{5,3}, e^{11,1}_{5,4}\right\} \\ +q^{\frac{1}{2}}\left\{e^{3,2}_{6,1}, e^{4,2}_{7,1}, e^{5,2}_{8,1}, e^{4,3}_{9,1}, e^{5,3}_{10,1}, e^{5,4}_{11,1}\right\} \\ +\overline{\mathbf{q}}^{\boldsymbol{\frac{1}{2}}}\left\{e^{6,1}_{2,3}, e^{7,1}_{2,4}, e^{8,1}_{2,5}, e^{9,1}_{3,4}, e^{10,1}_{3,5}, e^{11,1}_{4,5}\right\} \\ -\overline{q}^{\frac{1}{2}}\left\{e^{2,3}_{6,1}, e^{2,4}_{7,1}, e^{2,5}_{8,1}, e^{3,4}_{9,1}, e^{3,5}_{10,1}, e^{4,5}_{11,1}\right\}\end{array}\right\},$$

$$\Delta q^{3\alpha+3} A_1\left\{q\left\{\begin{array}{l} e^{7,3}_{9,2}, e^{8,3}_{10,2}, e^{8,4}_{11,2}, e^{10,4}_{11,3} \\ e^{9,2}_{7,3}, e^{10,2}_{8,3}, e^{11,2}_{8,4}, e^{11,3}_{10,4}\end{array}\right\}, -\left\{\begin{array}{l} e^{6,4}_{9,2}, e^{6,5}_{10,2}, e^{7,5}_{11,2}, e^{9,5}_{11,3} \\ e^{9,2}_{6,4}, e^{10,2}_{6,5}, e^{11,2}_{7,5}, e^{11,3}_{9,5}\end{array}\right\}, \overline{q}\left\{\begin{array}{l} e^{6,4}_{7,3}, e^{6,5}_{8,3}, e^{7,5}_{8,4}, e^{9,5}_{10,4} \\ e^{7,3}_{6,4}, e^{8,3}_{6,5}, e^{8,4}_{7,5}, e^{10,4}_{9,5}\end{array}\right\}\right\},$$

$$\Delta q^{3\alpha+\frac{1}{2}} A_0^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{-q^{\frac{1}{2}}\left\{\begin{array}{l} e^{2,9}_{12,1}, e^{2,10}_{13,1}, e^{2,11}_{14,1}, e^{3,11}_{15,1} \\ e^{12,1}_{2,9}, e^{13,1}_{2,10}, e^{14,1}_{2,11}, e^{15,1}_{3,11}\end{array}\right\}, +\overline{q}^{\frac{1}{2}}\left\{\begin{array}{l} e^{3,7}_{12,1}, e^{3,8}_{13,1}, e^{4,8}_{14,1}, e^{4,10}_{15,1} \\ e^{12,1}_{3,7}, e^{13,1}_{3,8}, e^{14,1}_{4,8}, e^{15,1}_{4,10}\end{array}\right\}\right\},$$

$$-\Delta q^{3\alpha+3} A_0^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{\begin{array}{l} e^{4,6}_{12,1}, e^{5,6}_{13,1}, e^{5,7}_{14,1}, e^{5,9}_{15,1} \\ e^{12,1}_{4,6}, e^{13,1}_{5,6}, e^{14,1}_{5,7}, e^{15,1}_{5,9}\end{array}\right\},$$

$$\Delta q^{4\alpha+3} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{q}^{\boldsymbol{\frac{1}{2}}}\left\{e^{7,6}_{12,2}, e^{8,6}_{13,2}, e^{9,6}_{12,3}, e^{10,6}_{13,3}, e^{8,7}_{14,2}, e^{9,7}_{12,4}, e^{11,7}_{14,4}, e^{10,8}_{13,5}, e^{11,8}_{14,5}, e^{10,9}_{15,3}, e^{11,9}_{15,4}, e^{11,10}_{15,5}\right\} \\ +q^{\frac{1}{2}}\left\{e^{12,2}_{7,6}, e^{13,2}_{8,6}, e^{12,3}_{9,6}, e^{13,3}_{10,6}, e^{14,2}_{8,7}, e^{12,4}_{9,7}, e^{14,4}_{11,7}, e^{13,5}_{10,8}, e^{14,5}_{11,8}, e^{15,3}_{10,9}, e^{15,4}_{11,9}, e^{15,5}_{11,10}\right\} \\ +\overline{\mathbf{q}}^{\boldsymbol{\frac{1}{2}}}\left\{e^{6,7}_{12,2}, e^{6,8}_{13,2}, e^{6,9}_{12,3}, e^{6,10}_{13,3}, e^{7,8}_{14,2}, e^{7,9}_{12,4}, e^{7,11}_{14,4}, e^{8,10}_{13,5}, e^{8,11}_{14,5}, e^{9,10}_{15,3}, e^{9,11}_{15,4}, e^{10,11}_{15,5}\right\} \\ -\overline{q}^{\frac{1}{2}}\left\{e^{12,2}_{6,7}, e^{13,2}_{6,8}, e^{12,3}_{6,9}, e^{13,3}_{6,10}, e^{14,2}_{7,8}, e^{12,4}_{7,9}, e^{14,4}_{7,11}, e^{13,5}_{8,10}, e^{14,5}_{8,11}, e^{15,3}_{9,10}, e^{15,4}_{9,11}, e^{15,5}_{10,11}\right\}\end{array}\right\},$$

$$\Delta q^{4\alpha+5} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{\begin{array}{l} q^{\frac{3}{2}}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{11,6}_{13,4}, e^{10,7}_{14,3}, e^{8,9}_{15,2}\right\} \\ +1\left\{e^{13,4}_{11,6}, e^{14,3}_{10,7}, e^{15,2}_{8,9}\right\}\end{array}\right\}, q^{\frac{1}{2}}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{11,6}_{12,5}, e^{7,10}_{15,2}, e^{9,8}_{14,3}\right\} \\ -1\left\{e^{12,5}_{11,6}, e^{15,2}_{7,10}, e^{14,3}_{9,8}\right\}\end{array}\right\} \\ \overline{q}^{\frac{3}{2}}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{6,11}_{14,3}, e^{7,10}_{13,4}, e^{8,9}_{12,5}\right\} \\ -1\left\{e^{14,3}_{6,11}, e^{13,4}_{7,10}, e^{12,5}_{8,9}\right\}\end{array}\right\}, \overline{q}^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,11}_{15,2}, e^{10,7}_{12,5}, e^{9,8}_{13,4}\right\} \\ +1\left\{e^{15,2}_{6,11}, e^{12,5}_{10,7}, e^{13,4}_{9,8}\right\}\end{array}\right\}\end{array}\right\},$$

$$\Delta^2 q^{4\alpha+4} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{q^{\frac{1}{2}}\left\{\begin{array}{l} +\mathbf{1}\left\{e^{8,9}_{14,3}, e^{10,7}_{13,4}\right\} \\ -1\left\{e^{14,3}_{8,9}, e^{13,4}_{10,7}\right\}\end{array}\right\}, \overline{q}^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{8,9}_{13,4}, e^{7,10}_{14,3}\right\} \\ +1\left\{e^{13,4}_{8,9}, e^{14,3}_{7,10}\right\}\end{array}\right\}\right\},$$

$$\Delta q^{4\alpha+\frac{11}{2}}(q^2-\overline{q}^2) A_1^{\frac{1}{2}} A_2^{\frac{1}{2}}\left\{q\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{11,6}}_{\mathbf{15,2}} \\ -e^{15,2}_{11,6}\end{array}\right\}, -\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{10,7}}_{\mathbf{15,2}}, +\mathbf{e}^{\mathbf{11,6}}_{\mathbf{14,3}} \\ -e^{15,2}_{10,7}, -e^{14,3}_{11,6}\end{array}\right\}, \overline{q}\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{9,8}}_{\mathbf{15,2}} \\ -e^{15,2}_{9,8}\end{array}\right\}\right\},$$

$$\Delta^2 q^{3\alpha+3} A_0^{\frac{1}{2}} A_1 A_2^{\frac{1}{2}}\left\{q\left\{\begin{array}{l} -\mathbf{1}\left\{e^{9,2}_{12,1}, e^{10,2}_{13,1}, e^{11,2}_{14,1}, e^{11,3}_{15,1}\right\} \\ +1\left\{e^{12,1}_{9,2}, e^{13,1}_{10,2}, e^{14,1}_{11,2}, e^{15,1}_{11,3}\right\}\end{array}\right\}, \left\{\begin{array}{l} +\mathbf{1}\left\{e^{7,3}_{12,1}, e^{8,3}_{13,1}, e^{8,4}_{14,1}, e^{10,4}_{15,1}\right\} \\ -1\left\{e^{12,1}_{7,3}, e^{13,1}_{8,3}, e^{14,1}_{8,4}, e^{15,1}_{10,4}\right\}\end{array}\right\}, \overline{q}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{6,4}_{12,1}, e^{6,5}_{13,1}, e^{7,5}_{14,1}, e^{9,5}_{15,1}\right\} \\ +1\left\{e^{12,1}_{6,4}, e^{13,1}_{6,5}, e^{14,1}_{7,5}, e^{15,1}_{9,5}\right\}\end{array}\right\}\right\},$$

$$\Delta q^{5\alpha+6} A_2\left\{-\overline{q}\left\{\begin{array}{l} e^{12,8}_{13,7}, e^{12,10}_{13,9}, e^{12,11}_{14,9}, e^{13,11}_{14,10} \\ e^{13,7}_{12,8}, e^{13,9}_{12,10}, e^{14,9}_{12,11}, e^{14,10}_{13,11}\end{array}\right\}, \left\{\begin{array}{l} e^{12,8}_{14,6}, e^{12,10}_{15,6}, e^{12,11}_{15,7}, e^{13,11}_{15,8} \\ e^{14,6}_{12,8}, e^{15,6}_{12,10}, e^{15,7}_{12,11}, e^{15,8}_{13,11}\end{array}\right\}, -q\left\{\begin{array}{l} e^{14,6}_{13,7}, e^{13,9}_{15,6}, e^{14,9}_{15,7}, e^{14,10}_{15,8} \\ e^{13,7}_{14,6}, e^{15,6}_{13,9}, e^{15,7}_{14,9}, e^{15,8}_{14,10}\end{array}\right\}\right\},$$

$$\Delta^2 q^{4\alpha+6} A_1 A_2\left\{-\overline{q}^2\left\{\begin{array}{l} e^{12,5}_{13,4} \\ e^{13,4}_{12,5}\end{array}\right\}, +\overline{q}\left\{\begin{array}{l} e^{12,5}_{14,3} \\ e^{14,3}_{12,5}\end{array}\right\}, -\left\{\begin{array}{l} e^{12,5}_{15,2}, e^{13,4}_{14,3} \\ e^{15,2}_{12,5}, e^{14,3}_{13,4}\end{array}\right\}, +q\left\{\begin{array}{l} e^{15,2}_{13,4} \\ e^{13,4}_{15,2}\end{array}\right\}, -q^2\left\{\begin{array}{l} e^{15,2}_{14,3} \\ e^{14,3}_{15,2}\end{array}\right\}\right\},$$

$$\Delta q^{5\alpha+5} A_1^{\frac{1}{2}} A_3^{\frac{1}{2}}\left\{\overline{q}\left\{\begin{array}{l} e^{16,2}_{6,14}, e^{16,3}_{6,15}, e^{16,4}_{7,15}, e^{16,5}_{8,15} \\ e^{6,14}_{16,2}, e^{6,15}_{16,3}, e^{7,15}_{16,4}, e^{8,15}_{16,5}\end{array}\right\}, -\left\{\begin{array}{l} e^{16,2}_{7,13}, e^{16,3}_{9,13}, e^{16,4}_{9,14}, e^{16,5}_{10,14} \\ e^{7,13}_{16,2}, e^{9,13}_{16,3}, e^{9,14}_{16,4}, e^{10,14}_{16,5}\end{array}\right\}, q\left\{\begin{array}{l} e^{8,12}_{16,2}, e^{10,12}_{16,3}, e^{11,12}_{16,4}, e^{11,13}_{16,5} \\ e^{16,2}_{8,12}, e^{16,3}_{10,12}, e^{16,4}_{11,12}, e^{16,5}_{11,13}\end{array}\right\}\right\},$$

$$\Delta q^{6\alpha+7} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{q}^{\boldsymbol{\frac{1}{2}}}\left\{e^{16,6}_{13,12}, e^{16,7}_{14,12}, e^{16,8}_{14,13}, e^{16,9}_{15,12}, e^{16,10}_{15,13}, e^{16,11}_{15,14}\right\} \\ +q^{\frac{1}{2}}\left\{e^{13,12}_{16,6}, e^{14,12}_{16,7}, e^{14,13}_{16,8}, e^{15,12}_{16,9}, e^{15,13}_{16,10}, e^{15,14}_{16,11}\right\} \\ +\overline{\mathbf{q}}^{\boldsymbol{\frac{1}{2}}}\left\{e^{16,6}_{12,13}, e^{16,7}_{12,14}, e^{16,8}_{13,14}, e^{16,9}_{12,15}, e^{16,10}_{13,15}, e^{16,11}_{14,15}\right\} \\ -\overline{q}^{\frac{1}{2}}\left\{e^{12,13}_{16,6}, e^{12,14}_{16,7}, e^{13,14}_{16,8}, e^{12,15}_{16,9}, e^{13,15}_{16,10}, e^{14,15}_{16,11}\right\}\end{array}\right\},$$

$$\Delta^2 q^{5\alpha+6} A_1^{\frac{1}{2}} A_2 A_3^{\frac{1}{2}}\left\{\overline{q}\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,2}_{12,8}, e^{16,3}_{12,10}, e^{16,4}_{12,11}, e^{16,5}_{13,11}\right\} \\ +1\left\{e^{12,8}_{16,2}, e^{12,10}_{16,3}, e^{12,11}_{16,4}, e^{13,11}_{16,5}\right\}\end{array}\right\}, \left\{\begin{array}{l} +\mathbf{1}\left\{e^{16,2}_{13,7}, e^{16,3}_{13,9}, e^{16,4}_{14,9}, e^{16,5}_{14,10}\right\} \\ -1\left\{e^{13,7}_{16,2}, e^{13,9}_{16,3}, e^{14,9}_{16,4}, e^{14,10}_{16,5}\right\}\end{array}\right\}, q\left\{\begin{array}{l} -\mathbf{1}\left\{e^{16,2}_{14,6}, e^{16,3}_{15,6}, e^{16,4}_{15,7}, e^{16,5}_{15,8}\right\} \\ +1\left\{e^{14,6}_{16,2}, e^{15,6}_{16,3}, e^{15,7}_{16,4}, e^{15,8}_{16,5}\right\}\end{array}\right\}\right\},$$

$$\Delta^2 q^{4\alpha+5} A_0^{\frac{1}{2}} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}}\left\{\overline{q}^2\left\{\begin{array}{l} e^{16,1}_{6,11} \\ e^{6,11}_{16,1}\end{array}\right\}, -\overline{q}\left\{\begin{array}{l} e^{16,1}_{7,10} \\ e^{7,10}_{16,1}\end{array}\right\}, \left\{\begin{array}{l} e^{16,1}_{8,9}, e^{16,1}_{9,8} \\ e^{8,9}_{16,1}, e^{9,8}_{16,1}\end{array}\right\}, -q\left\{\begin{array}{l} e^{16,1}_{10,7} \\ e^{10,7}_{16,1}\end{array}\right\}, q^2\left\{\begin{array}{l} e^{16,1}_{11,6} \\ e^{11,6}_{16,1}\end{array}\right\}\right\},$$

$$\Delta q^{4\alpha+3} A_0^{\frac{1}{2}} A_3^{\frac{1}{2}}\left\{\overline{q}^{\frac{3}{2}}\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{2,15}} \\ -e^{2,15}_{16,1}\end{array}\right\}, \overline{q}^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{3,14}} \\ +e^{3,14}_{16,1}\end{array}\right\}, q^{\frac{1}{2}}\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{4,13}} \\ -e^{4,13}_{16,1}\end{array}\right\}, q^{\frac{3}{2}}\left\{\begin{array}{l} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{5,12}} \\ +e^{5,12}_{16,1}\end{array}\right\}\right\},$$

$$\Delta^3 q^{4\alpha+6} A_0^{\frac{1}{2}} A_1 A_2 A_3^{\frac{1}{2}}\left\{\overline{q}^{\frac{3}{2}}\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{12,5}} \\ -e^{12,5}_{16,1}\end{array}\right\}, \overline{q}^{\frac{1}{2}}\left\{\begin{array}{l} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{13,4}} \\ +e^{13,4}_{16,1}\end{array}\right\}, q^{\frac{1}{2}}\left\{\begin{array}{l} +\mathbf{e}^{\mathbf{16,1}}_{\mathbf{14,3}} \\ -e^{14,3}_{16,1}\end{array}\right\}, q^{\frac{3}{2}}\left\{\begin{array}{l} -\mathbf{e}^{\mathbf{16,1}}_{\mathbf{15,2}} \\ +e^{15,2}_{16,1}\end{array}\right\}\right\}.$$

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